AD-A159 705 ASYMPTOTIC ANALYSIS OF NON-LINEAR ELLIPTIC AND PARABOLIC SINGULAR PERTURBATIONS(U) KATHOLIEKE UNIV NIJMEGEN FACULTEIT DER WISKUNDE EN NATUURWETE.. 1/1

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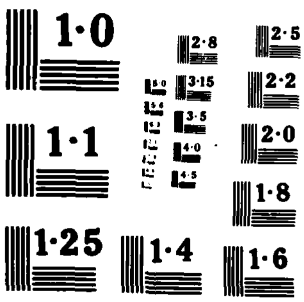
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AD-A159 705

ASYMPTOTIC ANALYSIS OF NON-LINEAR
ELLIPTIC AND PARABOLIC SINGULAR PERTURBATIONS

by

L.S. FRANK





CONTRACT NUMBER DAJA37-82-C-0731

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Faculteit W&N, KU, Nijmegen

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# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | |
|---|---|
| 1. REPORT NUMBER | 2. GOVT ACCESSION NO. AD-A159705 |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (and Subtitle) ASYMPTOTIC ANALYSIS OF NON-LINEAR ELLIPTIC AND PARABOLIC SINGULAR PERTURBATIONS. | 5. TYPE OF REPORT & PERIOD COVERED FINAL TECHNICAL REPORT SEPT '82 - SEPT '85 |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) L.S. FRANK | 8. CONTRACT OR GRANT NUMBER(s) DAJA37-82-C-0731 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS FACULTEIT DER WISKUNDE EN NATUURWETENSCHAPPNE, KATHOLIEKE UNIVERSITEIT, NIJMEGEN, THE NETHERLANDS | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102A 1T161102BH57-05 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS USARDSG-UK Box 65, FPO NY 09510-1500 | 12. REPORT DATE ~~July~~ Sep 1985 |
| | 13. NUMBER OF PAGES 50 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report) UNCLASSIFIED |
| | 15a. DECLASSIFICATION DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

PARTIAL DIFFERENTIAL OPERATORS, SINGULAR PERTURBATIONS, ASYMPTOTICS, VARIATIONAL CALCULUS, KINETIC THEORY, MEMBRANES WITH ENZYMOTIC ACTIVITY, MICHAELIS COEFFICIENT.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


SOME CLASSES OF NON-LINEAR SECOND ORDER ELLIPTIC AND PARABOLIC PARTIAL DIFFERENTIAL OPERATORS AFFECTED BY THE PRESENCE OF A SMALL PARAMETER ε ARE INVESTIGATED. THE REDUCED PROBLEM ( = 0) IS CHARACTERIZED BY THE APPEARANCE OF A FREE BOUNDARY OF THE SOLUTIONS. THE EXISTENCE, UNIQUENESS AND REGULARITY RESULTS ARE ESTABLISHED FOR BOTH PERTURBED AND REDUCED PROBLEMS. SHARP TWO-SIDED ESTIMATES FOR THE DIFFERENCE OF THE SOLUTIONS OF THE PERTURBED AND REDUCED PROBLEMS ARE PROVED AND SOME CONSTRUCTIVE


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PROCEDURES ARE FOUND OUT FOR LOCALIZING AND COMPUTING THE FREE BOUNDARY OF THE REDUCED PROBLEM. THE KINETIC THEORY OF MEMBRANES WITH ENZYMOTIC ACTIVITY IS ONE OF THE POSSIBLE FIELDS OF APPLICATIONS OF THE RESULTS ESTABLISHED, THE SMALL PARAMETER BEING THE SO-CALLED MICHAELIS' COEFFICIENT.

SUMMARY

Some classes of Non-linear Second Order Elliptic and Parabolic Partial Differential Operators affected by the presence of a small parameter $\varepsilon$ are investigated.

The reduced problem ($\varepsilon = 0$) is characterized by the appearence of a free boundary of the solutions.
The Existence, Uniqueness and regularity results are establised for both perturbed and reduced problems. Sharp two-sided estimates for the difference of the solutions of the perturbed and reduced problems are proved and some constructive procedures are found out for localizing and computing the free boundary of the reduced problem. The Kinetic Theory of membranes with enzymotic activity is one of the possible fields of applications of the results established, the small parameter being the so-called Michaelis' coefficient.

A-1

LIST OF KEYWORDS

Partial Differential Operators, Singular Perturbations, Asymptotics, Variational Calculus, Kinetic theory, membranes with enzymotic activity, Michaelis coefficient.

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L.S. Frank & W.D. Wendt

# Elliptic and Parabolic Singular Perturbations in the Kinetic Theory of Enzymes

## Introduction

In the Kinetic theory of membranes with enzymotic activity, a second order semilinear parabolic operator appears to be a suitable mathematical model for describing the dynamical process for the corresponding concentrations. The nonlinearity of this operator is affected by the presence of two positive parameters: $\varepsilon$ , the so-called Michaelis' constant, and $\lambda$ , the latter being connected with the ratio of initial concentrations of the enzyme and the substratum. For several realistic membranes, $\varepsilon$ is small compared to $\lambda$ and the data of the problem ([12]).

For each $\varepsilon > 0$ fixed, the mathematical model above fits the classical framework of Fréchet differentiable nonlinear operators. One can view this model as a family of perturbations (regular or singular, according to the magnitude of $\lambda$ ) of some reduced parabolic operator with a piecewise constant discontinuous nonlinearity. The same is also true for the corresponding elliptic stationary problem.

Further, for the stationary problem, there exists a critical value $\lambda_c$ of the second parameter $\lambda$ , such that for $\lambda < \lambda_c$ the original problem is a regular perturbation of the "reduced" one, whereas for $\lambda > \lambda_c$ , it becomes a singular perturbation and is characterized by the presence of boundary layers located in a neighbourhood of the free boundary of the solution to the "reduced" problem. The set $E_c$ of zeroes of the corresponding critical solution $u^{\lambda_c}$ plays an important role in the investigation of the reduced problem.

One of the central results presented here is the sharp error estimate in the $H_1$-norm for the difference of the solutions to the perturbed and reduced problems. Further, $\lambda_c$ is investigated as a functional of the data of the reduced problem.

Now the contents of the paper will be briefly sketched.
Part I deals with the stationary problem. Section 1.1 contains existence, uniqueness and regularity results as well as continuity results for the operators considered. In section 1.2, asymptotic solutions (for $\varepsilon \downarrow 0$ ) of the perturbed stationary problem are constructed under some regularity

assumptions on the free boundary and the sharp $H_1$-estimate for the difference $w_\varepsilon$ between the solutions of the reduced and the perturbed problems is established. Namely, the following two-sided estimate holds with some constant $C$ : $C^{-1}\varepsilon^{3/4} \leq \|w_\varepsilon\|_{H_1} \leq C\varepsilon^{3/4}$ . The special case of a piecewise linear nonlinearity plays an important role and allows a considerable simplification in the construction of asymptotic solutions in the general case, as well. A one-sided estimate of the form $\|w_\varepsilon\|_{H_1} \leq C\varepsilon^{1/2}$ , whose proof was merely based on the monotonicity of the nonlinearity and did not require regularity assumptions on the free boundary, had previously been given in [2], (see also [1(]).

In section 1.3, properties of the critical value $\lambda_c$ are stated and an improved convergence result of the form $\|w_\varepsilon\|_{H_1} \leq C\varepsilon(\lambda_c - \lambda)^{-1}$ is established when $\lambda < \lambda_c$ . Sections 1.4 and 1.5 contain investigation of $\lambda_c$ as a functional of the data.

Estimates of $\lambda_c$ from above and from below are indicated and the formula for the Frèchet derivative of $\lambda_c$ stated in [6,8] is proved. In section 1.6, the asymptotic behaviour for $\lambda \to \infty$ of the solution of the reduced problem and of the corresponding free boundary are investigated.

In Part II , the nonstationary problem is considered.

Section 2.1 contains existence, uniqueness and regularity results. In section 2.2 it is shown that the solution of the stationary problem is asymptotically stable in $H_1$ uniformly with respect to $\varepsilon$ . In section 2.3, an estimate for the difference of the solutions to the nonstationary perturbed and reduced problems is proved. Sections 2.4 and 2.5 deal with nonnegative solutions and special solutions of the Cauchy problem for the reduced equation.

Several results in this paper have been announced in [6-8 ].

The list of references contains essentially papers which are (to the best of the authors' knowledge) tightly connected with the topics presented here. We refer to [15 , 14 , 4] , [3] , [5 , 11] , [1 , 13 , 10] and [16 , 18 - 20] for more information concerning variational inequalities, maximal monotone operators, variational calculus and convex analysis, free boundary problems and singular perturbation theory.

## 0. Notation. Statement of the problem

Let $U \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$-boundary $\partial U$ and let $\mathbb{R}_+ = (0,\infty)$. Denote,

$$(0.1) \qquad Q = U \times \mathbb{R}_+ \ , \ Q_T = U \times (0,T) \ , \ \Gamma = \partial U \times \mathbb{R}_+ \ , \ \Gamma_T = \partial U \times [0,T) \ .$$

Let the function $f \in C^0(\mathbb{R},\mathbb{R})$ be piecewise continuously differentiable: $f \in C^1\left([s_k,s_{k+1}]\right) \quad \forall k \in \{0,\dots,r-1\}$, where $-\infty = s_0 < s_1 < \dots < s_r = \infty$. It is also assumed that

$$(0.2) \qquad \begin{cases} f(0) = 0 \\ 0 \le f'(s) \le L(1+s^2)^{-1} \quad \forall s \in \mathbb{R} \setminus \{s_1,\dots,s_{r-1}\}, \end{cases}$$

where $L > 0$ is constant.

As a consequence of (0.2), $f(s)$ is monotonically increasing on $\mathbb{R}$ and, moreover, there exist the limits

$$\lim_{s \to \pm\infty} f(s) = f_{\pm\infty}, \quad -\infty < f_{-\infty} \le f_{+\infty} < +\infty \ .$$

$H(s)$ being Heaviside's function, we associate with $f(s)$ the following function

$$(0.3) \qquad f_0(s) = f_{+\infty}H(s) + f_{-\infty}H(-s) \ , \quad \forall s \in \mathbb{R} \setminus \{0\} \ , \quad f_0(0) = 0 \ .$$

We also denote by $F(s)$ and $F_0(s)$ the primitive functions of $f(s)$ and $f_0(s)$ normalized by the condition: $F(0) = 0$, $F_0(0) = 0$. Let $\left(a_{kj}(x,t)\right)_{1\le k,j\le n}$ be uniformly with respect to $(x,t) \in \overline{Q}$ positive definite, and let $a_{kj}(x,t) \in C^\infty(\overline{Q})$. It is assumed that the family

$$(0.4) \qquad t \to A\left(x,t,\frac{\partial}{\partial x}\right) = -\sum_{1\le k,j\le n} \frac{\partial}{\partial x_j} a_{kj}(x,t)\frac{\partial}{\partial x_k} \ , \quad a_{kj} \in C^\infty(\overline{Q}) \ ;$$

stabilizes, as $t \to +\infty$, to the operator:

$$(0.5) \qquad A_\infty\left(x,\frac{\partial}{\partial x}\right) = -\sum_{1\le k,j\le n} \frac{\partial}{\partial x_j} a^\infty_{kj}(x)\frac{\partial}{\partial x_k} \ , \quad a^\infty_{kj} \in C^\infty(\overline{U}) \ .$$

The following initial-boundary value problem $\mathfrak{A}^\lambda_\varepsilon$ is considered:

$$
(0.6)\quad \begin{cases} \dfrac{\partial u_\varepsilon^\lambda}{\partial t} + A\left(x,t,\dfrac{\partial}{\partial x}\right)u_\varepsilon^\lambda + \lambda f\left(\dfrac{u_\varepsilon^\lambda}{\varepsilon}\right) = g(x,t) \ , & (x,t) \in Q \\ u_\varepsilon^\lambda(x,0) = \psi(x) \ , & x \in \bar{U} \\ \pi_0 u_\varepsilon^\lambda(x',t) = \phi(x',t) \ , & (x',t) \in \Gamma \end{cases}
$$

where $\lambda, \varepsilon$ are positive parameters, $\pi_0$ is the restriction operator to $\Gamma$, the data is supposed to have the following regularity:

$$
g \in C^0(\bar{Q}) \ , \quad \psi \in C^2(\bar{U}) \ , \quad \phi \in C^{2,1}(\Gamma) \ ,
$$

to satisfy the compatibility condition:

$$
(0.7)\qquad \pi_0\psi(x') = \phi(x',0) \ , \quad \forall x' \in \partial U
$$

and to stabilize to

$$
g_\infty \in C^0(\bar{U}) \ , \quad \phi_\infty \in C^2(\partial U) \ ,
$$

as $t \to +\infty$.

Here, as usual, $C^0(\bar{Q})$ is the space of all continuous in $\bar{Q}$ real-valued functions, $C^2(\bar{U})$ is the space of all twice continuously differentiable real-valued functions in $\bar{U}$ and $C^{2,1}(\Gamma)$ is the space of all continuous real-valued functions on $\Gamma$ such that their first derivatives with respect to $(x',t) \in \Gamma$ and the second derivatives with respect to $x' \in \partial U$ are continuous functions. The parameter $\varepsilon$ is assumed to be small compared to $\lambda$ : $0 < \varepsilon << \lambda$.

Along with the problem $\mathcal{O}\!\mathcal{L}_\varepsilon^\lambda$, we also consider the corresponding stationary problem $\mathcal{O}\!\mathcal{L}_{\varepsilon,\infty}^\lambda$ :

$$
(0.8)\quad \begin{cases} A_\infty\left(x,\dfrac{\partial}{\partial x}\right)u_{\varepsilon,\infty}^\lambda + \lambda f\left(\dfrac{u_{\varepsilon,\infty}^\lambda}{\varepsilon}\right) = g_\infty(x) \ , & x \in U \\ \pi_0 u_{\varepsilon,\infty}^\lambda(x') = \phi_\infty(x') & x' \in \partial U \ . \end{cases}
$$

<u>Example 0.1.</u> With $A = -\Delta$, $f(s) = s(1+|s|)^{-1}$, the problem (0.6) appears in the kinetic theory of membranes with enzymotic activity ([12]).

In this case $f_0(s) = \operatorname{sgn} s \ \forall s \in \mathbb{R} \setminus \{0\}$. Since in applications one is essentially interested in non-negative solutions ($u_\varepsilon^\lambda$ is interpreted as the

dynamical concentration of enzyme in this case), one can also define $f(s)$ to be $f(s) = s_+(1+s)^{-1}$ with $s_+ = \max(s,0)$ . Then one has $f_0(s) = H(s)$ .

Example 0.2. Let $1, a_+, a_- \in \mathbb{R}_+$ be fixed and let $f_1$ denote the piece-wise linear function

$$f_1(s) = H(s) \min\{ls, a_+\} + H(-s) \max\{ls, -a_-\}$$

An asymptotic (for $\varepsilon \downarrow 0$) solution of $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ with $f = f_1$ is constructed in section 1.2 below and used in order to investigate the asymptotic behaviour (for $\varepsilon \downarrow 0$) of the solution of $\mathcal{H}^{\lambda}_{\varepsilon,\infty}$ with general $f$ .

For a given function $u \in C^0(\overline{Q})$ (or $u \in C^0(\overline{U})$ ), denote by $E_+(u)$ , $E_-(u)$ the sets where $u > 0$ , $u = 0$ , $u < 0$ , respectively, whereas $\chi_+(u)$ , $\chi_0(u)$ , $\chi_-(u)$ stand for the characteristic functions of these sets.

We associate with $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon}$ the following "reduced" problem $\mathcal{O}\!\mathcal{L}^{\lambda}$ :

$$(0.9)\begin{cases} \dfrac{\partial u^\lambda}{\partial t} + A\left(x,t,\dfrac{\partial}{\partial x}\right)u^\lambda + \lambda f_0(u^\lambda) = g(x,t)\left[1-\chi_0(u^\lambda)\right] , & (x,t) \in Q \\ \lambda f_{-\infty} \le g(x,t) \le \lambda f_{+\infty} , & (x,t) \in \operatorname{int} E_0(u^\lambda) \\ u^\lambda(x,0) = \psi(x) , & x \in \overline{U} \\ \pi_0 u^\lambda(x',t) = \phi(x',t) , & (x',t) \in \Gamma . \end{cases}$$

The solution $u^\lambda$ of (0.9) is supposed to be continuous in $\overline{Q}$ , the differential equation and the condition $\lambda f_{-\infty} \le g(x,t) \le \lambda f_\infty$ in (0.9) are interpreted in the sense of Schwartz's distributions.

The corresponding stationary "reduced" problem $\mathcal{O}\!\mathcal{L}^{\lambda}_{\infty}$ is stated as follows:

$$(0.10)\begin{cases} A_\infty\left(x,\dfrac{\partial}{\partial x}\right)u^\lambda_\infty + \lambda f_0(u^\lambda_\infty) = g_\infty(x)\left[1-\chi_0(u^\lambda_\infty)\right] , & x \in U \\ \lambda f_{-\infty} \le g_\infty(x) \le \lambda f_{+\infty} , & x \in \operatorname{int} E_0(u^\lambda_\infty) \\ \pi_0 u^\lambda_\infty(x') = \phi_\infty(x') , & x' \in \partial U \end{cases}$$

where again $u^\lambda_\infty \in C^0(\overline{U})$ , the differential equation and the condition $\lambda f_{-\infty} \le g_\infty(x) \le \lambda f_\infty$ in (0.10) are interpreted in the distributional sense. The reduced problems (0.9),(0.10) can be reformulated in terms of maximal monotone operators (see [3]).

## I. Stationary problem

### 1. General properties of the operators considered

Both $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ and $\mathcal{O}\!\mathcal{L}^{\lambda}_{\infty}$ can be equivalently reformulated as variational minimization problems (see, for instance, [5]), where the corresponding functionals

$$D^{\lambda}_{\varepsilon}(u) = \int_U \left[\frac{1}{2} \sum_{1\le k,j\le n} a^{\infty}_{kj}(x)\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j} + \lambda\varepsilon F\left(\frac{u}{\varepsilon}\right) - g_{\infty}u\right]dx$$

$$D^{\lambda}(u) = \int_U \left[\frac{1}{2} \sum_{1\le k,j\le n} a^{\infty}_{kj}(x)\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j} + \lambda F_0(u) - g_{\infty}u\right]dx$$

are lower semi-continuous, coercive and strictly convex on the hyperplane

$$\Pi_{\Phi_\infty} = \left\{u \in H_1(U) \;\middle|\; u - \Phi_\infty \in \overset{\circ}{H}_1(U)\right\}$$

Here, as usual, $H_1(U)$ is the Sobolev space of order 1 and $\overset{\circ}{H}_1(U)$ is the subspace of those functions in $H_1(U)$, for which traces on $\partial U$ vanish; further $\Phi_\infty$ is the solution of the following boundary value problem:

$$(1.1.1)\qquad \begin{cases} A_\infty\left(x,\frac{\partial}{\partial x}\right)\Phi_\infty = g_\infty(x)\ , & x \in U \\ \pi_0\, \Phi_\infty(x') = \phi_\infty(x')\ , & x' \in \partial U\ . \end{cases}$$

Using the equivalent variational reformulation of $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$, $\mathcal{O}\!\mathcal{L}^{\lambda}_{\infty}$ and the classical a priori estimates for linear second order elliptic operators, one gets the following result:

<u>Theorem 1.1.1.</u> There exist well-defined solutions

$$u^{\lambda}_{\varepsilon,\infty} \in C^{1,\alpha}(\bar{U})\ , \quad u^{\lambda}_{\infty} \in C^{1,\alpha}(\bar{U})\ , \quad \forall\alpha \in [0,1)$$

of the problems $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ and $\mathcal{O}\!\mathcal{L}^{\lambda}_{\infty}$, respectively.
We use the same notation $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ for the operator

$$(1.1.2)\qquad \mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty} : H_2(U) \to L_2(U) \times H_{3/2}(\partial U)$$

associated with the boundary value problem (0.8), where, as usual, $H_s(U)$ and $H_r(\partial U)$ stand for the Sobolev spaces (of orders $s$ and $r$, respectively)

of functions in $U$ and on $\partial U$. We are going to state several results concerning the continuity properties of the nonlinear operator $\mathfrak{A}^{\lambda}_{\varepsilon,\infty}$ and its inverse.

<u>Proposition 1.1.2.</u> For any given $\varepsilon > 0$, the mapping (1.1.2) is a Lipschitz-continuous homeomorphism.

<u>Proposition 1.1.3.</u> For any given $\phi_\infty \in H_{1/2}(\partial U)$, the operator

$$(\mathfrak{A}^{\lambda}_{\varepsilon,\infty})^{-1} : H_{-1}(U) \to \Pi_{\phi_\infty}$$

is Lipschitz-continuous uniformly with respect to $\varepsilon$.

<u>Proposition 1.1.4.</u> With $u^{\lambda}_{\varepsilon,\infty}$ the solution of $\mathfrak{A}^{\lambda}_{\varepsilon,\infty}$, the mapping

$$\overline{\mathbb{R}}_+ \ni \lambda \to u^{\lambda}_{\varepsilon,\infty} \in H_1(U)$$

is Lipschitz-continuous uniformly with respect to $\varepsilon \in (0,\varepsilon_0]$, thus also for $\varepsilon = 0$.

The proofs of Propositions 1.1.2 - 1.1.4 essentially use the fact that $f(s)$ is monotonically increasing.

<u>Proposition 1.1.5.</u> For any given $g \in H_{-1}(U)$, the operator

$$H_{1/2}(\partial U) \ni \phi_\infty \xrightarrow{(\mathfrak{A}^{\lambda}_{\varepsilon,\infty})^{-1}} u^{\lambda}_{\varepsilon,\infty} \in H_1(U) \tag{1.1.3}$$

is Hölder-continuous with exponent $\alpha = \frac{1}{2}$ uniformly with respect to $\varepsilon$. Moreover, the following a priori estimate holds:

$$\begin{aligned} &\left\| (\mathfrak{A}^{\lambda}_{\varepsilon,\infty})^{-1}(\phi_1,g) - (\mathfrak{A}^{\lambda}_{\varepsilon,\infty})^{-1}(\phi_2,g)\right\|_{H_1(U)} \le \\ &\le C\left(\|\Phi_1 - \Phi_2\|^{1/2}_{L_2(U)} + \|\Phi_1 - \Phi_2\|_{H_1(U)}\right), \end{aligned} \tag{1.1.4}$$

where $\Phi_j$, $j = 1,2$, is the solution of the linear problem

$$\begin{cases} A_\infty \Phi_j = 0, & x \in U, \\ \pi_0 \Phi_j = \phi_j, & x' \in \partial U \end{cases} \tag{1.1.5}$$

and the constant $C$ does not depend on $\phi_j, g$ and $\varepsilon$.

<u>Proof.</u> Let $u_j = (\mathcal{O}\!\!\!\lambda_{\varepsilon,\infty})^{-1}(\phi_j, g)$ and denote $v_j = u_j - \Phi_j$, where $\Phi_j$ is defined by (1.1.5). Then $v = v_1 - v_2$ is the solution of the problem:

$$
(1.1.6)\qquad \begin{aligned} & A_\infty v + \lambda\left[f\left(\frac{v_1+\Phi_1}{\varepsilon}\right) - f\left(\frac{v_2+\Phi_2}{\varepsilon}\right)\right] = 0\,, && x \in U \\ & \pi_0 v(x') = 0 && x' \in \partial U \end{aligned}
$$

The inner product in $L^2(U)$ after the integration by part yields:

$$
\int_U \sum a_{kj}^{\infty}(x) v_{x_j} v_{x_k} dx + \lambda \int_U \left[f\left(\frac{v_1+\Phi_1}{\varepsilon}\right) - f\left(\frac{v_2+\Phi_2}{\varepsilon}\right)\right](\Phi_1 - \Phi_2 + v)dx =
$$

$$
= \lambda \int_U \left[f\left(\frac{v_1+\Phi_1}{\varepsilon}\right) - f\left(\frac{v_2+\Phi_2}{\varepsilon}\right)\right](\Phi_1 - \Phi_2)dx
$$

The monotonicity of $f(s)$ leads to the inequality:

$$
(1.1.7)\qquad \begin{aligned} & \int_U \sum a_{kj}^{\infty}(x) v_{x_j} v_{x_k} dx \le \lambda \int_U \left[f\left(\frac{u_1}{\varepsilon}\right) - f\left(\frac{u_2}{\varepsilon}\right)\right](\Phi_1 - \Phi_2)dx \le \\ & \le 2\lambda \max\left\{f_{+\infty}, |f_{-\infty}|\right\}(\text{meas } U)^{1/2} \|\Phi_1 - \Phi_2\|_{L^2(U)} \end{aligned}
$$

As a consequence of (1.1.7), one gets the estimate:

$$
\|\nabla v\|^2_{L_2(U)} \le 2\lambda\mu^{-1} \max\left\{f_{+\infty}, |f_{-\infty}|\right\}(\text{meas } U)^{1/2} \|\Phi_1 - \Phi_2\|_{L^2(U)}
$$

where $\mu$ is the ellipticity constant for $A_\infty\left(x, \frac{\partial}{\partial x}\right)$. Hence,

$$
(1.1.8)\qquad \|v\|_{H_1(U)} \le C_1 \|\Phi_1 - \Phi_2\|^{1/2}_{L^2(U)}
$$

where $C_1$ depends only on $\lambda, \mu, f_{\pm\infty}$, meas $U$ and the constant $\nu$ in the Poincarè's inequality: $\|\nabla v\|_{L^2(U)} \ge \nu \|v\|_{L^2(U)}$, $\forall v \in \overset{\circ}{H}_1(U)$.
As a consequence of (1.1.8), one gets (1.1.4) with $C = C_1 + 1$.

1.2. Convergence for $\varepsilon \to 0$

In this section, an estimate for the $H_1$-norm of $u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty$ is given which is a slight generalization of the result established previously in [16,2]. Moreover, if the free boundary $\partial E_0(u^\lambda_\infty)$ is a sufficiently smooth manifold, it is proved that the $H_1$-norm of $u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty$ is of order $O(\varepsilon^{3/4})$ as $\varepsilon \downarrow 0$. It will be shown in the next section that there exists a critical value $\lambda_c(\phi_\infty, g_\infty)$ of $\lambda$ such that for $\lambda < \lambda_c$, the norm of $u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty$ is of order $O(\varepsilon)$ as $\varepsilon \downarrow 0$.

Theorem 1.2.1. Under the regularity assumption

(1.2.1) $$\phi_\infty \in C^2(\partial U), \quad g_\infty \in C^0(\bar{U}),$$

the following estimate holds:

(1.2.2) $$\left\| u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty \right\|_{H_1(U)} \le C\varepsilon^{1/2},$$

where the constant $C$ depends only on $\lambda, f, A_\infty$ and $U$.

Theorem 1.2.1. is proved by using the monotonicity of $f$ and the fact that the functions $u^\lambda_{\varepsilon,\infty}$, $u^\lambda_\infty$ can be characterized as solutions of corresponding elliptic variational inequalities.

Using a compactness argument and the uniqueness of the solution $u^\lambda_\infty$ to the reduced problem, one can also prove the following

Proposition 1.2.2. If $g_\infty \in C^0(\bar{U})$, $\phi_\infty \in C^2(\partial U)$, then one has:

(1.2.3) $$\lim_{\varepsilon \to 0} \left\| u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty \right\|_{C^{1,\alpha}(\bar{U})} = 0, \qquad \forall \alpha \in [0,1).$$

Now we are going to establish the main result in this section.

Theorem 1.2.3. Assume that $\phi_\infty > 0$ on $\partial U$, $g_\infty \equiv 0$ in $U$, that the free boundary $\partial E_0(u^\lambda_\infty)$ is a $C^4$-manifold of dimension $n-1$, and that the function $f$ satisfies the condition (0.2). Then the following estimate holds:

(1.2.4) $$\left\| u^\lambda_{\varepsilon,\infty} - u^\lambda_\infty \right\|_{H_1(U)} \le C\varepsilon^{3/4},$$

where the constant $C$ does not depend $\varepsilon$.

One constructs asymptotic solutions of the problem $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ with a specially chosen piecewise linear function $f = f_l$ in order to prove this theorem. More precisely, introducing the function

$$(1.2.5) \qquad f_l(s) = H(s)\min\left\{ls, f_\infty\right\} + H(-s)\max\left\{ls, f_{-\infty}\right\}$$

where $H(s)$ is Heaviside's function and the parameter $l > 0$ will be chosen later, one applies an appropriate modification of Vishik-Lyusternik's method ([19,20]; see also [16,18]) for constructing asymptotic solutions of $\mathcal{O}\!\mathcal{L}^{\lambda}_{\varepsilon,\infty}$ with $f(s) = f_l(s)$ and for establishing (1.2.4).

For $x$ in a sufficiently small neighbourhood of the free boundary $\partial E_0(u^\lambda_\infty)$, define $x' \in \partial E_0(u^\lambda_\infty)$, $\rho \in \mathbb{R}$ by the relations:

$$|x - x'| = \min_{y' \in \partial E_0(u^\lambda)} |x - y'| = \operatorname{dist}\left(x, \partial E_0(u^\lambda)\right) = |\rho|$$

$$\rho > 0 \quad \text{on} \quad E_+(u^\lambda)$$

$$\rho \le 0 \quad \text{on} \quad E_0(u^\lambda)\,.$$

If $x \in E_+(u^\lambda)$ and lies in the neighbourhood above (where the coordinates $(x', \rho)$ are well defined), then the operator $A_\infty$ can be rewritten as follows:

$$A_\infty\left(x, \frac{\partial}{\partial x}\right) = -\left(a(x)\frac{\partial^2}{\partial \rho^2} + \left(b(x) + c(x)\nabla'\right)\frac{\partial}{\partial \rho} + B(x, \nabla')\right)$$

where $\nabla'$ denotes the gradient with respect to $x' \in \partial E_0(u^\lambda)$ and where $B(x,\nabla')$ is a differential operator of second order with sufficiently smooth coefficients. Besides, the functions $a(x) > 0$, $b(x)$, $c(x)$ are sufficiently smooth, since the manifold $\partial E_0(u^\lambda)$ is supposed to be sufficiently smooth. Let $U_{\varepsilon,i} = \left\{x \in U \;\middle|\; |\rho| < 2\varepsilon^{1/3}\right\}$ and $U_{\varepsilon,e} = \left\{x \in U \;\middle|\; |\rho| > \varepsilon^{1/3}\right\}$ denote the interior and the exterior region, respectively. In $U_{\varepsilon,i}$, an asymptotic solution is sought in the form

$$(1.2.6) \qquad v_\varepsilon(x', \rho) = \sum_{2 \le j \le 3} \varepsilon^{j/2} v_j\left(x', \varepsilon^{-1/2}\rho\right),$$

where the functions $v_j$, $j = 2,3$, are solutions of the following boundary value problems on $\mathbb{R}$ with $x' \in \partial E_0(u^\lambda)$ playing the role of

a parameter:

$$(1.2.7)\quad \begin{cases} a(x')\dfrac{d^2}{d\zeta^2}v_2(x',\zeta) - \lambda f_1\Big(v_2(x',\zeta)\Big) = 0\ , & \zeta \in \mathbb{R} \\ v_2(x',\zeta) = o(1) & ,\quad \zeta \to -\infty \\ v_2(x',\zeta) = \Big(2a(x')\Big)^{-1}\lambda f_\infty \zeta^2 + O(1) & ,\quad \zeta \to +\infty \end{cases}$$

$$(1.2.8)\quad \begin{cases} a(x')\dfrac{d^2}{d\zeta^2}v_2(x',\zeta) - \lambda f_1'\Big(v_2(x',\zeta)\Big)v_3(x',\zeta) = \gamma_3(x',\zeta)\ , & \zeta \in \mathbb{R} \\ v_3(x',\zeta) = o(1) & ,\quad \zeta \to -\infty \\ v_3(x',\zeta) = \lambda f_\infty (3!)^{-1}\beta_3(x')\zeta^3 + O(1) & ,\quad \zeta \to +\infty \end{cases}$$

with

$$\gamma_3(x',\zeta) = -\lambda\zeta\Big(a(x')\Big)^{-1}a_\rho(x')f_1\Big(v_2(x',\zeta)\Big) - \Big(b(x') + c(x')\nabla'\Big)v_{2\zeta}$$
$$\beta_3(x') = -\Big(a(x')\Big)^{-1}a_\rho(x') + \Big(b(x') + c(x')\nabla'\Big)\Big(\Big(a(x')\Big)^{-1}\Big)$$

The solution of (1.2.7) is given by

$$v_2^{(1)}(x',\zeta) = \begin{cases} \dfrac{1}{1}e^{\left(\sqrt{\frac{\lambda f_\infty 1}{a(x')}}\zeta\right)-1} & ,\quad \zeta \le \sqrt{\dfrac{a(x')}{\lambda f_\infty 1}} \\ \dfrac{\lambda f_\infty}{2a(x')}\zeta^2 + \dfrac{1}{21} & ,\quad \zeta > \sqrt{\dfrac{a(x')}{\lambda f_\infty 1}} \end{cases}$$

and, $(d/d\zeta)v_2$ being a solution of the homogeneous differential equation in (1.2.18), the boundary value problem for $v_3$ can be solved using the variation of constants' method.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a function which is identically one on the interval $[-1,1]$ and the support of which is contained in $[-2,2]$. Let $z_\varepsilon$ be defined by

$$(1.2.9)\qquad z_\varepsilon(x) = \chi\Big(\varepsilon^{-1/3}\rho\Big)v_\varepsilon(x',\rho) + \Big(1 - \chi\Big(\varepsilon^{-1/3}\rho\Big)\Big)u_\infty^\lambda(x)$$

Obviously, this function satisfies the boundary condition $\pi_0 z_\varepsilon = \phi_\infty$ on $\partial U$.

Lemma 1.2.4. There exist constants $C$, $\varepsilon_0$, such that

$$\left\| A_\infty z_\varepsilon + \lambda f_1\Big(\frac{z_\varepsilon}{\varepsilon}\Big)\right\|_{L_2(U)} \le C\varepsilon^{5/6}$$

for $\varepsilon \in [0,\varepsilon_0]$.

Proof. We shall proceed by splitting the proof in several steps.

(i) For $x \in U_{\varepsilon,e}$, on has:

$$u^{\lambda}(x) \geq p\rho^2 \geq p\varepsilon^{2/3}$$

with a constant $p > 0$. Thus, $\varepsilon^{-1} z_{\varepsilon}(x) = \varepsilon^{-1} u^{\lambda}(x) \geq p\varepsilon^{-1/3}$ and $f_1\left(\varepsilon^{-1} z_{\varepsilon}(x)\right) \equiv 1$ for $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0$ sufficiently small. Thus, $A_{\infty} z_{\varepsilon} + \lambda f_1(\varepsilon^{-1} z_{\varepsilon}) \equiv 0$ for $\varepsilon \in (0,\varepsilon_0]$, $x \in U_{\varepsilon,e}$.

(ii) It will be shown that

$$\text{(1.2.10)} \qquad \left\| f_1\left(v_2 + \sqrt{\varepsilon} v_3\right) - \left(f_1(v_2) + \sqrt{\varepsilon} f_1'(v_2) v_3\right) \right\|_{L^2(U_{\varepsilon,i})} \leq C\varepsilon$$

where $v_j = v_j\left(x', \varepsilon^{-1/2}\rho\right)$. Without restriction of generality, one can assume that $\lambda = f_{\infty} = 1$. Since $f_1$ is piecewise linear, the function on the left hand side of (1.2.10) is zero if the interval $\left(v_2 + \sqrt{\varepsilon} v_3, v_2\right)$ does not contain $l^{-1}$. Let now

$$S_{\varepsilon} = \left\{ (x',\rho) \;\middle|\; v_2 + \sqrt{\varepsilon} v_3 < l^{-1} < v_2 \right\}.$$

If $(x',\rho) \in S_{\varepsilon}$, then $\varepsilon^{-1/2}\rho > \zeta_0(x') \overset{\text{def}}{=} \sqrt{l^{-1} a(x')}$ and

$$\frac{1}{2a(x')} \frac{\rho^2}{\varepsilon} + \frac{1}{2l} + \sqrt{\varepsilon} v_3 < l^{-1} < \frac{1}{2a(x')} \frac{\rho^2}{\varepsilon} + \frac{1}{2l}$$

Since $|v_3(\zeta)| = O(\zeta^3)$, $\zeta \to \infty$, one obtains

$$\rho^2 - C\rho^3 < \varepsilon \cdot \zeta_0(x')^2 < \rho^2$$

with a constant $C$. Thus, $\left|\rho - \sqrt{\varepsilon}\zeta_0(x')\right| \leq C\varepsilon$. Since $l v_2\left(x', \zeta_0(x')\right) = 1$, the following inequality hold: for $x \in S_{\varepsilon}$:

$$\left| f_1\left(v_2 + \sqrt{\varepsilon} v_3\right) - \left(f_1(v_2) + \sqrt{\varepsilon} f_1'(v_2) v_3\right) \right| = \left| l\left(v_2 + \sqrt{\varepsilon} v_3\right) - 1 \right| \leq C\varepsilon^{1/2}.$$

The last inequality yields (1.2.10), since the measure of $S_{\varepsilon}$ is of order $O(\varepsilon)$.

(iii) It will be shown that

$$\text{(1.2.11)} \qquad \left\| A_{\infty} z_{\varepsilon} + \lambda\left(f_1(v_2) + f_1'(v_2)\sqrt{\varepsilon} v_3\right) \right\|_{C^0(\overline{U}_{\varepsilon,i})} \leq C\varepsilon^{2/3}$$

Consider first the region $|\rho| < \varepsilon^{1/3}$, where $\chi \equiv 1$. One has

$$B(x,\nabla')z_\varepsilon = B(x,\nabla')\left(\varepsilon v_2 + \varepsilon^{3/2}v_3\right) = O(\varepsilon^{2/3})$$

Thus, with $\zeta = \varepsilon^{-1/2}\rho$, one has

$$Az_\varepsilon + \lambda\left(f_1(v_2) + \sqrt{\varepsilon} f_1'(v_2)v_3\right) = r(x,\varepsilon) + O(\varepsilon^{2/3}) ,$$

where

$$\begin{aligned}(1.2.12) \quad r(x,\varepsilon) &= -a(x')v_{2\zeta\zeta}(x',\zeta) + \lambda f_1(v_2) + \\ &\quad + \sqrt{\varepsilon}\Big(-a(x')v_{3\zeta\zeta}(x',\zeta) + \lambda f'(v_2)v_3 - \zeta a_\rho(x')v_{2\zeta\zeta}(x',\zeta) - \\ &\quad - \left(b(x') + c(x')\nabla'\right)v_{2\zeta}\left(x', \frac{\rho}{\sqrt{\varepsilon}}\right)\Big) \\ &= O(\varepsilon^{2/3})\end{aligned}$$

according to the construction of $v_j$.
In the region $\varepsilon^{1/3} < |\rho| < 2\varepsilon^{1/3}$, one obtains

$$Az_\varepsilon + \lambda\left(f_1(v_2) + \sqrt{\varepsilon} f_1'(v_2)v_3\right) = \sum_{0\le i\le 2} r_i(x,\varepsilon)$$

where $r_i$, $0 \le i \le 2$, are given by

$$\begin{aligned} r_o(x,\varepsilon) &= r(x,\varepsilon) - \left(a(x') + \rho a_\rho(x')\right)(1-\chi)\left(u_{\rho\rho} - \left(v_{2\zeta\zeta} + \sqrt{\varepsilon}v_{3\zeta\zeta}\right)\right) \\ &\quad - (b + c\nabla')(1-\chi)\left(u_\rho - \sqrt{\varepsilon}v_{2\zeta}\right) + \chi B\left(\varepsilon v_2 + \varepsilon^{3/2}v_3\right) + (1-\chi)Bu^\lambda \end{aligned}$$

$$\begin{aligned} r_1(x,\varepsilon) &= -2a(x')\varepsilon^{-1/3}\chi'(\varepsilon^{-1/3}\rho)\frac{\partial}{\partial\rho}\left(\varepsilon v_2 + \varepsilon^{3/2}v_3 - u^\lambda\right) \\ &\quad - (b + c\nabla')\varepsilon^{-1/3}\chi'(\varepsilon^{-1/3}\rho)\left(\varepsilon v_2 + \varepsilon^{3/2}v_3 - u^\lambda\right) \end{aligned}$$

$$r_2(x,\varepsilon) = -a(x)\varepsilon^{-2/3}\chi''(\varepsilon^{-1/3}\rho)\left(\varepsilon v_2 + \varepsilon^{3/2}v_3 - u^\lambda\right)$$

Using the boundary conditions for $v_j$ and the asymptotic expansion

$$u^\lambda(x',\rho) = \left(2a(x')\right)^{-1}\lambda f_\infty \rho^2 + \lambda f_\infty (3!)^{-1}\beta_3(x')\rho^3 + O(\rho^4) , \quad \rho \downarrow 0 ,$$

one checks easily that

$$\sup_{\varepsilon^{1/3}<\rho<2\varepsilon^{1/3}} \left|r_i(x,\varepsilon)\right| \le C\varepsilon^{2/3} , \quad i = 0,1,2 ,$$

where the constant $C$ does not depend upon $\varepsilon$. Thus (1.2.11) is proved.

(iv) Since $u^\lambda(x) \geq p\rho^2$ for $x \in E_+(u^\lambda)$ with a constant $p>0$, one can choose $\varepsilon_0$ so small that for $\forall\varepsilon \in (0,\varepsilon_0]$, $\forall x \in U_{\varepsilon,e}$, $f\left(\varepsilon^{-1}z_\varepsilon(x)\right) = f_\infty$ and $f\left(v_2\left(x',\varepsilon^{-1/2}\rho\right)+\varepsilon^{1/2}v_3\left(x',\varepsilon^{-1/2}\rho\right)\right) = f_\infty$ for $x \in U_{\varepsilon,i} \cap U_{\varepsilon,e}$. Thus

$$\begin{aligned}
&\left\|A_\infty z_\varepsilon(x) + \lambda f_1\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right\|_{L_2(U)} = \\
&= \left\|A_\infty z_\varepsilon(x) + \lambda f_1\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right\|_{L_2(U_{\varepsilon,i})} \\
&= \left\|A_\infty z_\varepsilon(x) + \lambda f_1\left(v_2+\varepsilon^{1/2}v_3\right)\right\|_{L_2(U_{\varepsilon,i})} \\
&\leq \left\|A_\infty z_\varepsilon + \lambda\left(f_1(v_2) + f_1'(v_2)\sqrt{\varepsilon}v_3\right)\right\|_{L_2(U_{\varepsilon,i})} + \\
&\quad + \lambda\left\|f_1(v_2) + f_1'(v_2)\varepsilon^{1/2}v_3 - f_1\left(v_2+\varepsilon^{1/2}v_3\right)\right\|_{L_2(U_{\varepsilon,i})} \\
&\leq C\varepsilon^{5/6},
\end{aligned}$$

as a consequence of (1.2.10), (1.2.11) and given that meas $(U_{\varepsilon,i}) = O(\varepsilon^{1/3})$, $\varepsilon \downarrow 0$.

Lemma 1.2.4 is proved. □

In order to prove Theorem 1.2.3 above, several auxiliary results will be needed.

<u>Lemma 1.2.5.</u> There exists a well defined value of the parameter $l \in (0,\infty)$, such that

$$I(l) = \int_{-\infty}^{\infty}\left(f\left(v_2^{(l)}(\zeta)\right) - f_1\left(v_2^{(l)}(\zeta)\right)\right)d\zeta = 0 \tag{1.2.13}$$

<u>Proof.</u> Let $v(\zeta) = v_2^{(1)}(\zeta)$. Then $v_2^{(l)}(\zeta) = l^{-1}v(l^{1/2}\zeta)$. Using the substitution $\eta = v(\sqrt{l}\zeta)$, one gets

$$\sqrt{l}I(l) = \int_0^\infty\left(f(l^{-1}\eta) - f_1(\eta)\right)\frac{d\eta}{v'(v^{-1}(\eta))}$$

The right hand side is a strictly decreasing function of $l \in (0,\infty)$, so that $I(l)$ has at most one zero. For $l$ sufficiently large, one has $f(l^{-1}\eta) - f_1(\eta) < 0$ $\forall\eta > 0$, so that $I(l) < 0$ for $l >> 1$. Now $I(l) > 0$ for $l$ sufficiently small. In fact, one has:

$$(1.2.14)\qquad \left| \int_1^{\infty}\Big(f(l^{-1}\eta) - f_1(\eta)\Big)\Big(v'\big(v^{-1}(\eta)\big)\Big)^{-1}d\eta \right|$$
$$= \left| \int_1^{\infty}\Big(f(l^{-1}\eta) - f_{\infty}\Big)\Big(v'\big(v^{-1}(\eta)\big)\Big)^{-1}d\eta \right|$$
$$\le Cl\int_1^{\infty}\eta^{-3/2}d\eta$$

with $C$ independent of $l$. Hence,

$$\sqrt{l}\,I(l) = \int_0^1\Big(f(l^{-1}\eta) - f_1(\eta)\Big)\Big(v'\big(v^{-1}(\eta)\big)\Big)^{-1}d\eta + O(1)\ , \text{ when } l\downarrow 0\ .$$

Therefore,

$$\lim_{l\downarrow 0}\sqrt{l}\,I(l) = \int_0^1\Big(f_{\infty} - f_1(\eta)\Big)\Big(v'\big(v^{-1}(\eta)\big)\Big)^{-1}d\eta > 0\ ,$$

and Lemma 1.2.5 is proved. □

We choose $l$ to be the zero of $I(l)$. Let $h(s) = f(s) - f_1(s)$, and for $a>0$, define $U_a = \left\{x \in U \,\middle|\, \mathrm{dist}\big(x, \partial E_0(u_{\infty}^{\lambda})\big) > a\right\}$ $R_a = \left\{\rho \in \mathbb{R} \,\middle|\, |\rho| > a\right\}$.
We choose $a>0$ so small that for $|\rho| < a$, the mapping $x \to (x',\rho)$ is a diffeomorphism.

Lemma 1.2.6. There exist constants $C,\varepsilon_0 > 0$, which do not depend upon $\varepsilon$ and such that for $\varepsilon \in (0,\varepsilon_0]$ holds:

(i) $\left\| h\big(\varepsilon^{-1}z_{\varepsilon}(x)\big)\right\|_{L^2(U_a)} \le C\varepsilon$

(ii) $\left\| h\Big(v_2\big(x',\varepsilon^{-1/2}\rho\big)\Big)\right\|_{L^2(\partial E_0\times\mathbb{R}_a)} \le C\varepsilon$

(iii) $\left\| h\Big(v_2\big(x',\varepsilon^{-1/2}\rho\big)\Big)\right\|_{H_{-1}(U\setminus U_a)} \le C\varepsilon^{3/4}$.

Proof. (i) For $x \in U_a \cap E_+(u_{\infty}^{\lambda})$, one has: $z_{\varepsilon}(x) = u^{\lambda}(x) \ge p > 0$, where $p$ does not depend on $\varepsilon, x$. The inequality $|h(s)| \le C(1+|s|)^{-1}$ and the fact that for $x \in U_a \cap E_0(u_{\infty}^{\lambda})$, one has: $z_{\varepsilon}(x) \equiv 0$, so that $h\big(\varepsilon^{-1}z_{\varepsilon}(x)\big) = 0$ yield the first part of the Lemma.
(ii) Since for $\rho \in \mathbb{R}_a$ holds $v_2\big(x',\varepsilon^{-1/2}\rho\big) \ge \varepsilon^{-1}p$, where $p$ does not depend on $x',\varepsilon$, the second inequality can be proved similarly to the first one.

(iii) Denote $t(x',s) = h\left(v_2(x',s)\right)$. One has

$$\left\| t\left(x',\varepsilon^{-1/2}\rho\right)\right\|_{H_{-1}(\partial E_o\times(-a,0))} \le \tag{1.2.15}$$

$$\le \left\| t\left(x',\varepsilon^{-1/2}\rho\right)\right\|_{H_{-1}(\partial E_o\times\mathbb{R})} + \left\| t\left(x',\varepsilon^{-1/2}\rho\right)\right\|_{L^2(\partial E_o\times\mathbb{R}_a)}$$

Let $T(x',\zeta) = \int_{-\infty}^{\zeta} t(x',s)ds$. The inequality $|t(x',s)| \le C(1+s^2)^{-1}$, which holds uniformly with respect to $x' \in \partial E_o$, and Lemma 1.2.5 yield $|T(x',\zeta)| \le C(1+|\zeta|)^{-1}$. Thus, $T \in L^2(\partial E_o\times\mathbb{R})$ and

$$\left\| t\left(x',\varepsilon^{-1/2}\rho\right)\right\|_{H_{-1}(\partial E_o\times\mathbb{R})} = \sup_{\varphi\in H_1(U)} \|\varphi\|^{-1}_{H_1(U)} \left|\int t\left(x',\varepsilon^{-1/2}\rho\right)\varphi(x',\rho)dx'd\rho\right|$$

$$= \sup_{\varphi\in H_1} \|\varphi\|^{-1}_{H_1(U)} \left|\int \varepsilon^{1/2} T\left(x',\varepsilon^{-1/2}\rho\right)\frac{\partial\varphi}{\partial\rho}(x',\rho)dx'd\rho\right|$$

$$\le \varepsilon^{1/2}\left\| T\left(x',\varepsilon^{-1/2}\rho\right)\right\|_{L^2(\partial E_o\times\mathbb{R})} \sup_{\varphi\in H_1(U)} \|\varphi\|^{-1}_{H_1(U)} \left\|\frac{\partial\varphi}{\partial\rho}\right\|_{L^2(U)}$$

$$\le C\varepsilon^{3/4}$$

As a consequence of (ii), the second term on the right hand side of (1.2.15) is of order $O(\varepsilon)$ and that ends the proof of Lemma 1.2.6.

□

Lemma 1.2.7. (i) There exist constants $C, \varepsilon_o$, such that

$$C^{-1}\left|v_2(\varepsilon^{-1/2}\rho)\right| \le \varepsilon^{-1}|z_\varepsilon(x)| \le C\left|v_2(\varepsilon^{-1/2}\rho)\right| \quad \forall\rho\in(0,a) \quad \forall\varepsilon\in(0,\varepsilon_o]. \tag{1.2.16}$$

(ii) There exists a constant $C>0$ such that

$$\left\|h\left(\varepsilon^{-1}z_\varepsilon(x)\right) - h\left(v_2(\varepsilon^{-1/2}\rho)\right)\right\|_{L^2(U\setminus U_a)} \le C\varepsilon^{3/4} \quad \forall\varepsilon\in(0,\varepsilon_o]. \tag{1.2.17}$$

Proof. (i) For $0<\rho<\varepsilon^{1/3}$, one has

$$\varepsilon^{-1}z_\varepsilon(x) = v_2(\varepsilon^{-1/2}\rho)\left(1+\left(v_2(\varepsilon^{-1/2}\rho)\right)^{-1}\varepsilon^{1/2}v_3(\varepsilon^{-1/2}\rho)\right)$$

The inequality

$$\left|v_2(\varepsilon^{-1}\rho)^{-1}\varepsilon^{1/2}v_3(\varepsilon^{-1/2}\rho)\right| \le C\rho \le C\varepsilon^{1/3}$$

implies that (1.2.16) holds for $0 < \rho < \varepsilon^{1/3}$ . Now let $\varepsilon^{1/3} < \rho < 2\varepsilon^{1/3}$ . Then

$$\varepsilon^{-1} z_\varepsilon(x) = v_2(\varepsilon^{-1/2}\rho)\Big(1 + r_1(\varepsilon, x)\Big)$$

where the function

$$r_1(\varepsilon, x) = v_2(\varepsilon^{-1/2}\rho)^{-1}\Big[\chi(\varepsilon^{-1/3}\rho)\varepsilon^{1/2}v_3 + \Big(1 - \chi(\varepsilon^{-1/3}\rho)\Big)\Big(\varepsilon^{-1}u^\lambda(x) - v_2(\varepsilon^{-1/2}\rho)\Big)\Big]$$

can be estimated as follows:

$$\big|r_1(\varepsilon, x)\big| \le C\rho \le 2C\varepsilon^{1/3} . \quad \forall \varepsilon \in (0, \varepsilon_0] ,$$

with some constant $C > 0$ .

Finally, let $2\varepsilon^{1/3} < \rho < a$ . Then

$$\begin{aligned}\varepsilon^{-1} z_\varepsilon(x) = \varepsilon^{-1}u^\lambda(x) &= \varepsilon^{-1}\Big(\Big(2a(x')\Big)^{-1}\rho^2 + O(\rho^3)\Big) \\ &= v_2(\varepsilon^{-1/2}\rho)\Big(1 + O\Big(\Big(v_2(\varepsilon^{-\frac{1}{2}}\rho)\Big)^{-1}\Big)\Big)\end{aligned}$$

and (1.2.16) is proved.

(ii) The left hand side of (1.2.17) can be estimated as follows:

$$\begin{aligned}&\Big\| h\Big(\varepsilon^{-1}z_\varepsilon(x)\Big) - h\Big(v_2(\varepsilon^{-1/2}\rho)\Big)\Big\|^2_{L^2(U\setminus U_a)} \le \\ &\le \int_{|\rho|<a}\Big(\sup_{\Theta\in(v_2,\varepsilon^{-1}z_\varepsilon)} |h'(\Theta)|\Big)^2\big|\varepsilon^{-1}z_\varepsilon(x) - v_2(\varepsilon^{-1/2}\rho)\big|^2 dx \\ &\le C\int_{0<\rho<a}\Big(\sup_{\Theta\in(v_2,\varepsilon^{-1}z_\varepsilon)} (1+\Theta^2)^{-1}\Big)^2\big|\varepsilon^{-1}z_\varepsilon(x) - v_2(\varepsilon^{-1/2}\rho)\big|^2 dx + \\ &\quad + C\int_{-a<\rho<0}\big|\varepsilon^{-1}z_\varepsilon(x) - v_2(\varepsilon^{-1/2}\rho)\big|^2 dx\end{aligned}$$

Using (1.2.16) and the asymptotic behaviour of $v_2(\zeta)$ for $\zeta \to \infty$ , one obtains:

$$\begin{aligned}&\Big\| h\Big(\varepsilon^{-1}z_\varepsilon(x)\Big) - h\Big(v_2(\varepsilon^{-1/2}\rho)\Big)\Big\|^2_{L^2(U\setminus U_a)} \le \\ &\le C\int_{0<\rho<a}\Big(1 + (\varepsilon^{-1/2}\rho)^4\Big)^{-2}\Big(1 + \varepsilon^{-1}\rho^3\Big)^2 d\rho + C\int_{-a<\rho<0}\varepsilon\big|v_3(\varepsilon^{-1/2}\rho)\big|^2 d\rho \\ &\le C\varepsilon^{3/2} \ \forall \varepsilon \in (0, \varepsilon_0] ,\end{aligned}$$

where $C$ does not depend upon $\varepsilon$, and that ends the proof of Lemma 1.2.7.

□

Proof of Theorem 1.2.3. Let $l \in (0,\infty)$ be the zero of the function $I(l)$ defined by (1.2.13), and let $z_\varepsilon(x)$ be given by (1.2.9). One has:

$$A_\infty z_\varepsilon + \lambda f_1\left(\varepsilon^{-1} z_\varepsilon(x)\right) = r_\varepsilon(x) ,$$

where, according to Lemma 1.2.4,

$$\|r_\varepsilon\|_{L_2(U)} \le C\varepsilon^{5/6} , \quad \forall \varepsilon \in (0,\varepsilon_0] , \tag{1.2.18}$$

with some constant $C > 0$ which does not depend on $\varepsilon$.

Writing the differential equations for $u_\varepsilon^\lambda$ and for $z_\varepsilon$ and taking the difference, one gets for $u_\varepsilon^\lambda - z_\varepsilon$ the following differential equation.

$$\begin{aligned} A_\infty(u_\varepsilon^\lambda - z_\varepsilon) + \lambda\left(f\left(\varepsilon^{-1}u_\varepsilon^\lambda(x)\right) - f\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right) &= \\ = -r_\varepsilon(x) - \lambda\left(f\left(\varepsilon^{-1}z_\varepsilon(x)\right) - f_1\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right) \end{aligned}$$

Taking the inner product with $u_\varepsilon^\lambda - z_\varepsilon$ in $L^2(U)$ in the last equation and using the monotonicity of $f$, one gets the following estimates:

$$\int_U (u_\varepsilon^\lambda - z_\varepsilon) A_\infty (u_\varepsilon^\lambda - z_\varepsilon)\,dx \le \left( \|r_\varepsilon\|_{H_{-1}(U)} + \lambda \left\| f\left(\varepsilon^{-1}z_\varepsilon(x)\right) - f_1\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right\|_{H_{-1}(U)}\right) \cdot \|u_\varepsilon^\lambda - z_\varepsilon\|_{H_1(U)}$$

The integration by part and Poincaré's Lemma yield:

$$\|u_\varepsilon^\lambda - z_\varepsilon\|_{H_1(U)} \le C\left(\|r_\varepsilon\|_{H_{-1}(U)} + \left\| f\left(\varepsilon^{-1}z_\varepsilon(x)\right) - f_1\left(\varepsilon^{-1}z_\varepsilon(x)\right)\right\|_{H_{-1}(U)}\right)$$

The last inequality is also a consequence of Proposition 1.1.3. According to (1.2.18), the first term on the right hand side is bounded by $C\varepsilon^{5/6}$.

According to the Lemmas 1.2.6, 1.2.7, the second term can be estimated as follows:

$$
\begin{aligned}
&\left\| f\left(\varepsilon^{-1} z_\varepsilon(x)\right) - f_1\left(\varepsilon^{-1} z_\varepsilon(x)\right)\right\|_{H_{-1}(U)} \\
&= \left\| h\left(\varepsilon^{-1} z_\varepsilon(x)\right)\right\|_{H_{-1}(U)} \\
&\le \left\| h\left(\varepsilon^{-1} z_\varepsilon(x)\right)\right\|_{H_{-1}(U_a)} + \left\| h\left(\varepsilon^{-1} z_\varepsilon(x)\right)\right\|_{H_{-1}(U\setminus U_a)} \\
&\le \left\| h\left(\varepsilon^{-1} z_\varepsilon(x)\right)\right\|_{L^2(U_a)} + \left\| h\left(v_2(x',\varepsilon^{-1/2}\rho)\right)\right\|_{H_{-1}(U\setminus U_a)} + \\
&\qquad + \left\| h\left(\varepsilon^{-1} z_\varepsilon(x)\right) - h\left(v_2(x',\varepsilon^{-1/2}\rho)\right)\right\|_{H_{-1}(U\setminus U_a)} \\
&\le C\varepsilon^{3/4} .
\end{aligned}
$$

This ends the proof of Theorem 1.2.3.

□

Remark 1.2.8. Consider the problem $\mathcal{O}\!\mathcal{l}^{\lambda}_{\varepsilon,\infty}$ in $U = (-1,1) \subset \mathbb{R}$ with $f = f_1$ defined in (1.2.5) and $A_\infty = -(\frac{d}{dx})^2$, $g_\infty \equiv 0$, $\phi_\infty(x') = 1$. The function $u^\lambda_\infty(x) = (\lambda/2)(|x| - \xi)^2_+$, $\xi = 1 - (2\lambda^{-1})^{1/2}$, is the solution of the reduced problem $\mathcal{O}\!\mathcal{l}^{\lambda}_{\infty}$. Let $v_2$ be the solution of (1.2.7) and let

$$
(1.2.19) \qquad z_\varepsilon(x) = u^\lambda_\infty(x)\left(1 - \chi(|x| - \xi)\right) + \varepsilon v_2\left(\sqrt{\lambda/\varepsilon}(|x| - \xi)\right)\chi(|x| - \xi) .
$$

Similarly to the proof of Lemma 1.2.4, one checks that

$$
(1.2.20) \qquad \left\| -z''_\varepsilon + \lambda f_1(\varepsilon^{-1} z_\varepsilon)\right\|_{C^0(\overline{U})} \le C\varepsilon
$$

where the constant $C$ does not depend on $\varepsilon$. Further, $\pi_o z_\varepsilon = 1$. Partial integration yields:

$$
(1.2.21) \qquad \left\| u^\lambda_{\varepsilon,\infty} - z_\varepsilon\right\|_{H_1(U)} \le C\varepsilon ,
$$

where the constant $C$ does not depend on $\varepsilon$.

Since $f_1'(+0) > 0$, the function $v_2(\zeta)$ and its derivative decrease exponentially for $\zeta \to -\infty$. Thus,

$$
(1.2.22) \qquad \gamma.\varepsilon^{3/4} \le \|z_\varepsilon\|_{H_1(-\xi,\xi)} \quad \forall \varepsilon \in (0,\varepsilon_o] ,
$$

where the constant $\gamma > 0$ does not depend on $\varepsilon$. The inequalities (1.2.21), (1.2.22) yield:

$$
\begin{aligned}
\left\| u^{\lambda}_{\varepsilon,\infty} - u^{\lambda}_{\infty} \right\|_{H_1(U)} &\geq \left\| u^{\lambda}_{\varepsilon,\infty} - u^{\lambda}_{\infty} \right\|_{H_1(-\xi,\xi)} \\
\geq \left\| u^{\lambda}_{\varepsilon,\infty} \right\|_{H_1(-\xi,\xi)} &\geq \| z_{\varepsilon} \|_{H_1(-\xi,\xi)} - \left\| u^{\lambda}_{\varepsilon,\infty} - z_{\varepsilon} \right\|_{H_1(U)} \\
&\geq \gamma \varepsilon^{3/4} - C\varepsilon \\
&\geq \frac{\gamma}{2} \varepsilon^{3/4} , \quad \forall \varepsilon \in (0,\varepsilon_0] ,
\end{aligned}
$$

where $\varepsilon_0 > 0$ is sufficiently small. Thus, one has the following two-sided error estimate:

(1.2.23) $$C^{-1}\varepsilon^{3/4} \leq \left\| u^{\lambda}_{\varepsilon,\infty} - u^{\lambda}_{\infty} \right\|_{H_1(U)} \leq C\varepsilon^{3/4}$$

with some constant $C > 0$ which does not depend on $\varepsilon$. It can be shown that the estimate (1.2.23) holds in the general case, as well, if the assumptions of Theorem 1.2.3 and the condition $f'(+0) > 0$ are satisfied.

<u>Remark 1.2.9.</u> If $g_{\infty} \not\equiv 0$, then the same argument with corresponding slight modifications in the construction of the asymptotic solutions of the problem $\mathcal{O}\!\mathcal{l}^{\lambda}_{\varepsilon,\infty}$ with $f = f_1$ leads to the same estimate under the assumptions of Theorem 1.2.3.

## 1.3. The critical value $\lambda_c$ of $\lambda$.

If $\phi_\infty(x') \neq 0 \ \forall x' \in \partial U$, then some critical value $\lambda_c$ of the parameter $\lambda$ plays a special role in the investigation of the boundary value problem $\mathfrak{A}^\lambda_\infty$. Namely, if $\phi_\infty > 0$, then for $\lambda < \lambda_c$ the problem $\mathfrak{A}^\lambda_\infty$ becomes linear, whereas for $\lambda > \lambda_c$ it is a nonlinear problem with piecewise constant discontinuous (across $\partial E_0(u^\lambda_\infty)$) nonlinearity. Denote by $C^0_+(\partial U)$ and $C^0_-(\partial U)$ the cones of continuous positive and negative functions on $\partial U$, respectively. Further, let $G(x,y)$ be Green's function for $A_\infty\left(x,\frac{\partial}{\partial x}\right)$ in $U$ with Dirichlet boundary conditions on $\partial U$ and denote by $E(x,y')$ the Poisson kernel for the Dirichlet problem for the equation $A_\infty u = 0$.

Theorem 1.3.1. (i) If $\phi_\infty \in C^0_+(\partial U)$ (respectively, $\phi_\infty \in C^0_-(\partial U)$), then there exists a well defined critical value $\lambda_c = \lambda^+_c(\phi_\infty,g_\infty)$ (respectively, $\lambda_c = \lambda^-_c(\phi_\infty,g_\infty)$), such that

$$E_0(u^\lambda_\infty) = \emptyset \quad \text{if} \quad \lambda < \lambda_c ,$$

meas $\left(E_0(u^\lambda_\infty) \cup E_-(u^\lambda_\infty)\right) > 0$ if $\lambda > \lambda_c$ (respectively, meas$\left(E_0(u^\lambda_\infty) \cup E_+(u^\lambda_\infty)\right) > 0$ if $\lambda > \lambda_c$)

(ii) If $\lambda_c \geq \lambda$, then $u^\lambda_\infty$ is the solution of the linear problem

$$\text{(1.3.1)} \qquad \begin{cases} A_\infty\left(x,\frac{\partial}{\partial x}\right)v^\lambda_\infty(x) = g_\infty(x) - \lambda f_\infty , & x \in U \\ \pi_0 v^\lambda_\infty(x) = \phi_\infty(x') , & x' \in \partial U \end{cases}$$

If $g_\infty \equiv 0$, then meas $E_0(u^{\lambda_c}_\infty) = 0$.

(iii) The functionals $\lambda^+_c$, $\lambda^-_c$ can be represented as follows:

$$\text{(1.3.2)} \qquad \lambda^\pm_c(\phi_\infty,g_\infty) = \min_{x\in U} \Lambda^\pm_{\phi_\infty,g_\infty}(x) ,$$

where the function $\Lambda^\pm_{\phi_\infty,g_\infty}$ is defined by:

$$\text{(1.3.3)} \qquad \Lambda^\pm_{\phi_\infty,g_\infty}(x) = \left(f_{\pm\infty}\int_U G(x,y)dy\right)^{-1}\left(\int_{\partial U} E(x,y')\phi_\infty(y')d\sigma_{y'} + \int_U G(x,y)g_\infty(y)dy\right)$$

The proof of Theorem 1.3.1 is similar to the proof of Theorem 2 in [9].

**Proposition 1.3.2.** The functional $(\phi_\infty, g_\infty) \to \lambda_c^+(\phi_\infty, g_\infty)$ has the following properties:

(i) $\lambda_c^+(\alpha\phi_\infty, \alpha g_\infty) = \alpha\lambda_c^+(\phi_\infty, g_\infty)$ , $\forall\alpha > 0$

(ii) $\lambda_c^+\left(\phi_\infty^{(1)}, g_\infty^{(1)}\right) \le \lambda_c^+\left(\phi_\infty^{(2)}, g_\infty^{(2)}\right)$ , if $0 \le \phi_\infty^{(1)} \le \phi_\infty^{(2)}$ , $g_\infty^{(1)} \le g_\infty^{(2)}$

(iii) $\lambda_c^+\left(\gamma\phi_\infty^{(1)} + (1-\gamma)\phi_\infty^{(2)}, \gamma g_\infty^{(1)} + (1-\gamma)g_\infty^{(2)}\right) \ge \gamma\lambda_c^+\left(\phi_\infty^{(1)}, g_\infty^{(1)}\right) + (1-\gamma)\lambda_c^+\left(\phi_\infty^{(2)}, g_\infty^{(2)}\right)$ $\forall\gamma \in [0,1]$.

Proof. One proves (i) - (iii), using the formula (1.3.2) for $\lambda_c^+(\phi_\infty, g_\infty)$ . □

Analogous properties has the functional $\lambda_c^-(\phi_\infty, g_\infty)$ .

For $\lambda < \lambda_c$ , the convergence result given in section 1.2 can be improved. One has the

**Theorem 1.3.3.** Assume that $\phi_\infty \in C_+^0(\partial U)$ and that $\lambda_c > \lambda$ . Then the following estimate holds:

$$\left\| u_{\varepsilon,\infty}^{\lambda} - u_\infty^{\lambda} \right\|_{W_{2,p}(U)} \le C\varepsilon(\lambda_c - \lambda)^{-1} , \quad \forall\varepsilon > 0 , \quad \forall p,\ 1 < p < \infty \tag{1.3.4}$$

where the constant $C$ depends only upon $p, L, U, A_\infty, \phi_\infty$ and $g_\infty$ .

Proof. Since $\mathcal{O}\!\mathcal{l}_\infty^{\lambda}$ for $\lambda < \lambda_c$ is linear, the function $w = u_{\varepsilon,\infty}^{\lambda} - u_\infty^{\lambda}$ is the solution of the problem:

$$\begin{aligned} A_\infty w(x) &= \lambda\left[f_{+\infty} - f\left(\frac{u_{\varepsilon,\infty}^{\lambda}}{\varepsilon}\right)\right] , && x \in U \\ \pi_0 w(x') &= 0 , && x' \in \partial U . \end{aligned} \tag{1.3.5}$$

One can write:

$$u_\infty^{\lambda}(x) = u^{\lambda_c}(x) + (\lambda_c - \lambda)v(x) , \quad \lambda \le \lambda_c ,$$

where $v(x)$ is the solution of the problem

$$\begin{aligned} A_\infty v(x) &= 1 , && x \in U \\ \pi_0 v(x') &= 0 , && x' \in \partial U \end{aligned} \tag{1.3.6}$$

Since $v(x) > 0$ , $\forall x \in U$ and $u^{\lambda_c}(x') > 0$ , $\forall x' \in \partial U$ , one can find a constant $\gamma > 0$ such that the sets:

$$U_1 = \left\{ x \in U \;\middle|\; u^{\lambda_c}(x) \geq \gamma(\lambda_c - \lambda) \right\} , \; U_2 = \left\{ x \in U \;\middle|\; v(x) \geq \gamma(\lambda_c - \lambda) \right\}$$

cover $\bar{U}$ , so that one has:

$$u_\infty^\lambda(x) \geq \gamma(\lambda_c - \lambda) \; , \quad \forall x \in \bar{U}$$

Of course, $\gamma$ depends upon $\phi_\infty$ and $g_\infty$ .

Denote $v_\varepsilon(x) = \dfrac{\partial u^\lambda_{\varepsilon,\infty}}{\partial \varepsilon}$ . Then $v_\varepsilon$ is the solution of the problem:

$$A_\infty v_\varepsilon + \frac{\lambda}{\varepsilon} f'\left(\frac{u^\lambda_{\varepsilon,\infty}}{\varepsilon}\right) v_\varepsilon = \frac{1}{\varepsilon^2} f'\left(\frac{u^\lambda_{\varepsilon,\infty}}{\varepsilon}\right) u^\lambda_{\varepsilon,\infty} \; , \quad x \in U$$

$$\pi_0 v_\varepsilon(x') = 0 \; , \qquad x' \in \partial U$$

Since $f'(s) \geq 0$ , $u^\lambda_{\varepsilon,\infty} \geq 0$ for $\varepsilon$ sufficiently small (because $u_\infty^\lambda > 0$ , $\forall x \in \bar{U}$ , $\forall \lambda < \lambda_c$ ), one gets the conclusion that $v_\varepsilon(x) \geq 0$ , so that $u^\lambda_{\varepsilon,\infty}$ is monotonically increasing function of $\varepsilon$ and, in particular, one has:

$$u^\lambda_{\varepsilon,\infty}(x) \geq u^\lambda_\infty(x) \geq \gamma(\lambda_c - \lambda) \; , \quad \forall x \in \bar{U} \; , \quad \varepsilon > 0$$

Hence,

$$0 \leq f_{+\infty} - f\left(\frac{u^\lambda_{\varepsilon,\infty}}{\varepsilon}\right) = \int_{\frac{u^\lambda_{\varepsilon,\infty}}{\varepsilon}}^{\infty} f'(s)\,ds \leq L \frac{\varepsilon}{\varepsilon + u^\lambda_{\varepsilon,\infty}} \leq \frac{L}{\gamma(\lambda_c - \lambda)} \, \varepsilon$$

and

(1.3.7) $$\left\| f_{+\infty} - f\left(\frac{u^\lambda_{\varepsilon,\infty}}{\varepsilon}\right) \right\|_{L_p(U)} \leq \frac{L}{\gamma} \, \varepsilon (\lambda_c - \lambda)^{-1} \, (\text{meas } U)^{1/p} \; .$$

As a consequence of the a priori estimates for second order linear elliptic operators, one gets, using (1.3.5), (1.3.6), the estimate (1.3.4).

<u>Corollary 1.3.4.</u> If $\lambda_c - \lambda = \varepsilon^\Theta$ , $0 < \Theta < 1$ , then $u^\lambda_{\varepsilon,\infty}$ converges to $u_\infty^{\lambda_c}$ in $W_{2,p}(U)$ $\forall p < \infty$ , as $\varepsilon \to 0$ ,

<u>Corollary 1.3.5.</u> If $\lambda_c - \lambda = \varepsilon^\Theta$ , $0 < \Theta < 1$ , then $u^\lambda_{\varepsilon,\infty}$ converges to $u_\infty^{\lambda_c}$ in $H_1(U)$ and the rate of convergence is $O(\varepsilon^\Theta)$ . In fact, let $\mu = \lambda_c - \lambda^{1-\Theta}$ . Then, using Proposition 1.1.4 and Theorem 1.3.3, one gets:

$$\left\| u_{\varepsilon,\infty}^{\lambda} - u_{\infty}^{\lambda_c} \right\|_1 \le \left\| u_{\varepsilon,\infty}^{\lambda} - u_{\varepsilon,\infty}^{\mu} \right\|_1 + \left\| u_{\varepsilon,\infty}^{\mu} - u_{\infty}^{\lambda_c} \right\|_1$$

$$\le C\left( |\lambda - \mu| + C\varepsilon (\lambda_c - \mu)^{-1} \right)$$

$$\le C\ \varepsilon^{\Theta}.$$

## 1.4. Estimates for the critical value of $\lambda$

Let $G(x,y)$ denote Green's function for the operator $A_\infty(x,\frac{\partial}{\partial x})$ with homogeneous Dirichlet boundary conditions and let $E(x,y')$ be the Poisson kernel for the Dirichlet problem for the equation $A_\infty u = 0$. Denote by $C_+^0(U)$ and $C_+^0(\partial U)$ the cone of positive continuous functions on $U$ and $\partial U$, respectively. For functions $g \in C_+^0(U)$ and $\phi \in C_+^0(\partial U)$, let $\mathcal{M}_t(g)$ and $\mathcal{M}_t(\phi)$ be their mean values of order t:

$$\mathcal{M}_t(g) = \left(\frac{1}{\text{meas}(U)} \int_U g(x)^t dx\right)^{\frac{1}{t}}, \quad \forall t \in \mathbb{R}$$

$$\mathcal{M}_t(\phi) = \left(\frac{1}{\text{meas}(\partial U)} \int_{\partial U} \phi(x')^t d\sigma_{x'}\right)^{\frac{1}{t}}, \quad \forall t \in \mathbb{R}$$

$$\mathcal{M}_t(\phi) = \left(\tfrac{1}{2} \sum_{x' \in \partial U} \phi(x')^t\right)^{\frac{1}{t}}, \quad \forall t \in \mathbb{R}, \text{ if } n = 1.$$

In this section, it is assumed that $g_\infty \in C_+^0(U)$, $\phi_\infty \in C_+^0(\partial U)$. For the investigation of the functional $\lambda_c(\phi_\infty, g_\infty)$, the assumption of positivity of $g_\infty$ is not a restriction of the generality. Indeed, (1.3.2) implies that:

$$\lambda_c^+(\phi_\infty, g_\infty) = \lambda_c^+(\phi_\infty, g_\infty+\rho) - (f_\infty)^{-1}\rho \qquad \forall \rho \in \mathbb{R}.$$

Let

(1.4.1.) $\quad v(x) = \int_U G(x,y)dy$

and let $\sum$ be the set of the points where the function v attains a global maximum. For functions $g \in C_+^0(U)$, $\phi \in C_+^0(\partial U)$, and for $x_0 \in \sum$, define the mean values $N_{x_0}(g)$, $N_{x_0}(\phi)$ as follows:

$$N_{x_0}(g) = (v(x_0))^{-1} \int_U G(x_0,y)g(y)dy,$$

$$N_{x_0}(\phi) = \int_{\partial U} E(x_0,y')\phi(y')d\sigma_{y'}.$$

One has the

Proposition 1.4.1. Let $U \subset \mathbb{R}^n$, $n \geq 2$. For $\forall(\phi_\infty, g_\infty) \in C_+^0(\partial U) \times C_+^0(U)$, $\forall t > 0$, the following estimate holds:

(1.4.2.) $$\rho M_{1-\frac{n}{2}}(\phi_\infty) + \rho_t M_{-t}(g_\infty) \leq \lambda_c^+(\phi_\infty, g_\infty) \leq$$
$$\leq (f_\infty)^{-1} \min_{x_0 \in \Sigma} (v(x_0)^{-1} N_{x_0}(\phi_\infty) + N_{x_0}(g_\infty))$$

where the constants $\rho, \rho_t$ are given by the formulae:

$$\rho = (f_\infty)^{-1} \mathrm{meas}(\partial U) \cdot \min_{x \in U} M_{\frac{n-2}{n}}(v(x)^{-1} E(x,\cdot))$$

$$\rho_t = (f_\infty)^{-1} \mathrm{meas}(U) \cdot \min_{x \in U} M_{\frac{t}{t+1}}(v(x)^{-1} G(x,\cdot)).$$

Proof. The second part of (1.4.2) is obtained by estimating the maximum in (1.3.3) by the value of $\Lambda^+_{\phi_\infty, g_\infty}$ at $x = x_0$.

In order to prove the first part of (1.4.2), note that for positive functions $h_1, h_2$, and for $p > 1$, $p^{-1} + q^{-1} = 1$, Hölder's inequality can be rewritten as follows:

(1.4.3.) $$\int h_1 \geq \left(\int h_1^{\frac{1}{p}} h_2\right)^p \left(\int h_2^q\right)^{-\frac{p}{q}}.$$

One has:

$$f_\infty \lambda_c^+(\phi_\infty, g_\infty) = \min_{x \in U} \Big((v(x)^{-1} \int_{\partial U} E(x,y')\phi_\infty(y')d\sigma_{y'} +$$
$$+ (v(x))^{-1} \int_U G(x,y) g_\infty(y)dy\Big)$$
$$\geq \min_{x \in U} \Big((v(x))^{-1} \int_{\partial U} E(x,y')\phi_\infty(y')d\sigma_{y'}\Big) +$$
$$+ \min_{x \in U} \Big((v(x))^{-1} \int_U G(x,y) g_\infty(y)dy\Big).$$

In order to estimate the first term from below, one applies (1.4.3) with $p = (n-2)^{-1} n$ and

$$h_1(y') = (v(x))^{-1} E(x,y')\phi_\infty(y'), \quad h_2(y') = \phi_\infty(y')^{-\frac{1}{p}}.$$

The second term can be estimated similarly, with $p = t^{-1}(t+1)$ and

$$h_1(y) = (v(x))^{-1}G(x,y)g_\infty(y), \quad h_2(y) = g_\infty(y)^{-\frac{1}{p}}.$$

This yields the claim (1.4.2). $\square$

For the rest of this section, U is assumed to be the unit ball in $\mathbb{R}^n$. If one has:

(1.4.4.) $A_\infty(x,\frac{\partial}{\partial x}) = -\Delta, \quad f_0(s) = \operatorname{sgn} s.$

(1.4.5.) $g_\infty(x) \equiv 0,$

the estimate (1.4.2) takes the following form:

(1.4.6.) $2n\ M_{1-\frac{n}{2}}(\phi_\infty) \le \lambda_c^+(\phi_\infty,0) \le 2n\ M_1(\phi_\infty).$

Indeed, in this case, the number $\rho$ can be computed as follows:

$$\rho = \min_{|x|<1} 2n(\Omega_n)^{-1}\int_{|y'|=1} |x-y'|^{2-n}d\sigma_{y'} = \min_{|x|<1} 2n = 2n \text{ if } n \ge 3$$

$$\rho = \min_{|x|<1} 4(\Omega_2)^{-1}\int_{|y'|=1} \ln|x-y'|d\sigma_{y'} = \min_{|x|<1} 4 = 2n \text{ if } n = 2$$

where $\Omega_n$ denotes the surface of the unit ball in $\mathbb{R}^n$.

The following result shows that the estimate (1.4.6) is sharp.

Proposition 1.4.2. (i) If $n \ge 2$, then there exist nonconstant function $\phi_j \in C_+^0(\partial U)$, $j = 1,2$, such that

$$\lambda_c^+(\phi_1,0) = 2n\ M_{1-\frac{n}{2}}(\phi_1), \quad \lambda_c^+(\phi_2,0) = 2n\ M_1(\phi_2).$$

(ii) There are no positive constants $\varepsilon_n$ and $\delta_n$ such that

$$2n\ M_{1-\frac{n}{2}+\varepsilon_n}(\phi) \le \lambda_c^+(\phi,0) \quad \forall\phi \in C_+^0(\partial U)$$

or $$\lambda_c^+(\phi,0) \le 2n\ M_{1-\delta_n}(\phi) \quad \forall\phi \in C_+^0(\partial U).$$

*Proof.* For $\xi \in U-\{0\}$ fixed, let $\phi_1(x') = |x'-\xi|^2$, $x' \in \partial U$ (which is, of course, the restriction to $\partial U$ of a linear function).
The corresponding critical solution is the function

$$u_\infty^{2n} = |x-\xi|^2,$$

such that:

$$(1.4.7.) \quad \lambda_c^+(\phi_1,0) = 2n = 2n\, M_{1-\frac{n}{2}}(\phi_1).$$

Let $\phi_2$ be the trace on $\partial U$ of the harmonic function $1+\frac{1}{2}x_1x_2$. The corresponding critical solution is given by the formula

$$u_\infty^{2n}(x) = |x|^2+\tfrac{1}{2}x_1x_2$$

such that $\lambda_c^+(\phi_2,0) = 2n = 2n\, M_1(\phi_2)$, where in the last step the mean value theorem for harmonic functions was used. The claim (ii) is an immediate consequence of (i) and of the following monotonicity property of the mean value: $M_{t_1}(\phi) < M_{t_2}(\phi)$ if $t_1 < t_2$ and $\phi$ is nonconstant on $\partial B_1$. $\square$ Consider now the one dimensional case $U = (-1,1)$.

*Proposition 1.4.3.* (i) Under the assumption (1.4.4), one has:

$$(1.4.8.) \quad 2M_{\frac{1}{2}}(\phi_\infty)+2e^{-1}M_0(g_\infty) \le \lambda_c^+(\phi_\infty,g_\infty) \le 2M_1(\phi_\infty)+\int_{-1}^{1}(1-|y|)g_\infty(y)dy.$$

(ii) Under the assumptions (1.4.4), (1.4.5), one has:

$$(1.4.9.) \quad \lambda_c^+(\phi_\infty,0) = 2M_{\frac{1}{2}}(\phi_\infty).$$

*Proof.* A direct computation shows that:

$$(1.4.10.) \quad \min_{|x|<1}\left((1-x^2)^{-1}\sum_{|x'|=1}(1+x'x)\phi_\infty(x')\right) = 2M_{\frac{1}{2}}(\phi_\infty).$$

This proves (1.4.9). The second inequality in (1.4.8) is obtained

similarly as the upper bound for $\lambda_0^+$ in (1.4.2). In order to prove the first inequality in (1.4.8), one uses (1.4.10) and Hölder's inequality:

$$\lambda_c^+(\phi_\infty, g_\infty) = \min_{|x|<1} 2(1-x^2)^{-1}(\tfrac{1}{2}\sum_{|x'|=1}(1+x'x)\phi_\infty(x') +$$
$$+ \int_{-1}^{1} G(x,y)g_\infty(y)dy)$$
$$\geq \min_{|x|<1} ((1-x^2)^{-1}\sum_{|x'|=1}(1+x'x)\phi_\infty(x')) +$$
$$+ \min_{|x|<1} 2(1-x^2)^{-1}\int_{-1}^{1} G(x,y)g_\infty(y)dy$$
$$\geq 2M_{\frac{1}{2}}(\phi_\infty) + \rho_t M_t(g_\infty) \qquad \forall t > 0$$

with $\rho_t$ the same as in Proposition 1.4.1. A computation shows that $\rho_t = 2\cdot(1+(t+1)^{-1}t)^{-t^{-1}(t+1)}$, such that $\lim_{t\downarrow 0}\rho_t = 2e^{-1}$. $\square$

As an extension of (1.4.9) to the multidimensional case one can mention the following fact: if $\phi_\infty(x')$ can be extended as a linear function on the ball, then $\lambda_c^+(\phi_\infty,0) = M_{1-n/2}(\phi_\infty)$. (See (1.4.7)). Thus, an explicit formula can be given for $\lambda_c^+(\phi_\infty,0)$ if $\phi_\infty$ can be extended as a linear function. It seems to be impossible to find such a formula for $\lambda_c^+(\phi_\infty,0)$ for $n > 1$ and $\phi_\infty \in C_+^0(\partial U)$ (see Remark 2 in [9]).

## 1.5. The critical set in the case of the Laplacian

It will be assumed that

(1.5.1.) $A_\infty(x,\frac{\partial}{\partial x}) = -\Delta, \quad f_\infty = 1.$

Denote

(1.5.2.) $E_c(\phi_\infty,g_\infty) \stackrel{\text{def}}{=} E_0(v_\infty^{\lambda_c^+(\phi_\infty,g_\infty)}) \quad \forall(\phi_\infty,g_\infty) \in C_+^0(\partial U) \times C^\alpha(U)$

where $v_\infty^\lambda$ is the solution of (1.3.1). (If $\lambda_c^+ \geq 0$, then the proof of Theorem 1.3.1 yields: $E_c(\phi_\infty,g_\infty) = E_0(u_\infty^{\lambda_c^+(\phi_\infty,g_\infty)})$).

Theorem 1.5.1. Let $(\phi_\infty,g_\infty) \in C_+^0(\partial U) \times C^\alpha(U)$ with $\alpha \in (0,1]$. Then for $\forall\xi \in E_c(\phi_\infty,g_\infty)$, there exists $\psi_\xi \in C_+^0(\partial U)$ such that:

(1.5.3.) $E_c(\phi_\infty+\delta\psi_\xi,g_\infty) = \{\xi\}, \quad \forall\delta > 0$

and, moreover, for the solution $v_\delta^{\lambda_c^+}$ of (1.3.1) corresponding to the data $\phi_\infty+\delta\psi_\xi,g_\infty$, the matrix of second derivatives $D_x^2 v_\delta(\xi)$ is positive definite, $\forall\delta > 0$.

Proof. Without loss of generality, it will be assumed that $f_\infty = 1$. Define the function $\psi_\xi \in C_+^0(\partial U)$ as follows:

(1.5.4.) $\psi_\xi(x') = |x'-\xi|^2, \quad \forall x' \in \partial U,\ \xi \in E_c(\phi_\infty,g_\infty).$

Denote by $w^\lambda$ the solution of $\mathcal{O}\!\mathcal{L}_\infty^\lambda$ with $g_\infty \equiv 0$ and the boundary condition $\pi_0 w^\lambda = \psi_\xi$. For $\lambda = \lambda_c^+(\psi_\xi,0)$, one finds easily the corresponding critical solution:

$$w^{\lambda_c^+}(x) = |x-\xi|^2 = \Psi_\xi(x)-2n\ v(x)$$

where $\Psi_\xi$ is the harmonic function in U, such that $\pi_0\Psi_\xi = \psi_\xi$ and v is

defined by (1.4.1). Hence,

$$\lambda_c^+(\psi_\xi) = \Delta w^{\lambda_c^+}(x) = 2n.$$

Further, since $w^{\lambda_c^+}(x) > 0 \quad \forall x \in \bar{U}\setminus\{\xi\}$, one has:

$$\text{(1.5.5.)} \quad \Lambda^+_{\psi_\xi,0}(x) \overset{\text{def}}{=} (v(x))^{-1}\Psi_\xi(x) > 2n = \Lambda^+_{\psi_\xi,0}(\xi) = \min_{y\in\bar{U}} \Lambda^+_{\psi_\xi,0}(y).$$

Let $u^0_\infty(x)$ denote the solution of the problem $\mathcal{Q}^\lambda$ for $\lambda = 0$. Using (1.5.5) and the fact that $\xi$ is a global minimum of $\Lambda^+_{\phi_\infty,g_\infty}$, one gets:

$$\begin{aligned}\Lambda^+_{\phi_\infty+\delta\psi_\xi,g_\infty}(x) &\overset{\text{def}}{=} (v(x))^{-1}(u^0_\infty(x)+\delta\Psi_\xi(x)) \\ &> (v(x))^{-1}u^0_\infty(x)+\delta(v(\xi))^{-1}\Psi_\xi(\xi) \geq \\ &\geq (v(\xi))^{-1}(u^0_\infty(\xi)+\delta\Psi_\xi(\xi)) = \Lambda^+_{\phi_\infty+\delta\psi_\xi,g_\infty}(\xi), \\ &\qquad \forall x \in \bar{U}\setminus\{\xi\}.\end{aligned}$$

Therefore, $\xi \in U$ is the only point where the function $\Lambda_{\phi_\infty+\delta\psi_\xi,g_\infty}(x)$ attains its minimum. As a consequence, the function

$$v_1(x) = u^0_\infty(x)-\lambda^+_c(\phi_\infty+\delta\psi_\xi,g_\infty)\ v(x)$$

is such that $E_0(v_1) = \{\xi\}$.

Now the second claim of Theorem 1.5.1 will be proved. A straightforward computation using the relations

$$v_\delta^{\lambda_c}(\xi) = 0, \quad \nabla_x v_\delta^{\lambda_c}(\xi) = 0$$

yields the formula:

$$\text{(1.5.6.)} \quad D^2\Lambda^+_{\phi_\infty+\delta\psi_\xi,g_\infty}(\xi) = (v(\xi))^{-1}D^2v_\delta^{\lambda_c}(\xi), \quad \forall \xi \in E_0(u_\delta^{\lambda_c}).$$

Hence,

$$\begin{aligned}D^2v_\delta^{\lambda_c}(\xi) &= v(\xi)D^2\Lambda^+_{\phi_\infty+\delta\psi_\xi,g_\infty}(\xi) \\ &= v(\xi)(D^2\Lambda^+_{\phi_\infty,g_\infty}(\xi)+\delta D^2\Lambda^+_{\psi_\xi,0}(\xi))\end{aligned}$$

$$= v(\xi) D^2 \Lambda^+_{\phi_\infty, g_\infty}(\xi) + \delta D^2_x(|x-\xi|^2)\,|_{x=\xi}$$

$$\geq \delta\ 2n\ \mathrm{Id}.$$

where Id denotes the identity matrix. $\square$

Now we are going to prove that for $g_\infty \equiv 0$ and for a large class of boundary functions $\phi_\infty$, the set $E_c(\phi_\infty, 0)$ consists of only one point.

<u>Definition 1.5.2</u>. For $\alpha \in (0,1]$, $O^\alpha_+$ is defined to be the set of the pairs $(\phi_\infty, g_\infty) \in C^0_+(\partial U) \times C^\alpha(\bar{U})$, such that the critical set $E_c(\phi_\infty, g_\infty)$ contains only one point $\xi$ and such that, for the solution $v^{\lambda_c}_\infty(\phi_\infty, g_\infty)$ of (1.3.1), the Hessian $D^2 v^{\lambda_c}_\infty(\xi)$ is positive definite at $\xi$.

As a consequence of Theorem 1.5.1, the set $O^\alpha_+$ is dense in $C^0_+(\partial U) \times C^\alpha(\bar{U})$.

<u>Theorem 1.5.3</u>. $O^\alpha_+$ is open in $C^0_+(\partial U) \times C^\alpha(\bar{U})$.

<u>Proof</u>. Let $(\phi_\infty, g_\infty) \in O^\alpha_+$ and $\{\xi\} = E_c(\phi_\infty, g_\infty)$.
With $\Lambda^+_{\phi_\infty, g_\infty}$ defined by (1.3.3), one gets as in (1.5.6):

(1.5.8.) $D^2_x \Lambda^+_{\phi_\infty, g_\infty}(\xi) = (v(\xi))^{-1} D^2_x v^{\lambda_c}_\infty(\xi) \geq 2\gamma\ \mathrm{Id}$

where $\gamma > 0$. Thus, there exists a constant $\delta_1 > 0$ such that

(1.5.9.) $D^2_x \Lambda^+_{\phi_\infty, g_\infty}(x) \geq \gamma\ \mathrm{Id}, \quad \forall x \in B_{\delta_1} = \{x \in U \mid |x-\xi| < \delta_1\}.$

Since $\xi$ is the only point in the critical set and therefore the only global minimum of $\Lambda^+_{\phi_\infty, g_\infty}(x)$, one can choose a constant $\delta_2 > 0$ such that

(1.5.10.) $\Lambda^+_{\phi_\infty, g_\infty}(x) \geq \Lambda^+_{\phi_\infty, g_\infty}(\xi) + \delta_2, \quad \forall x \in \bar{U} \backslash B_{\delta_1}.$

The inequalities (1.5.9), (1.5.10) will be used in order to show that there exists a constant $\rho > 0$ such that for any $(\psi_\infty, h_\infty) \in C^0_+(\partial U) \times C^\alpha(\bar{U})$, satisfying the condition

$$[\psi_\infty-\phi_\infty]_{C^0(\partial U)}+[h_\infty-g_\infty]_{C^\alpha(\overline{U})}\le\rho,$$

the function $\Lambda^+_{\phi_\infty,h_\infty}$ has still only one global minimum. Indeed, one has as a consequence of (1.3.3)

$$(1.5.11.)\quad \lambda^+_c(\psi_\infty,h_\infty)=\min_{x\in U}\Lambda^+_{\psi_\infty,h_\infty}(x)\le\Lambda^+_{\psi_\infty,h_\infty}(\xi)\le\Lambda^+_{\phi_\infty,g_\infty}(\xi)+\rho(1+(v(\xi))^{-1}).$$

Since

$$\begin{aligned}\Lambda^+_{\psi_\infty,h_\infty}(x)&\ge(v(x))^{-1}(\min\psi_\infty+v(x)\min h_\infty)\\&\ge(v(x))^{-1}(\min\phi_\infty-\rho)+g_\infty-\rho,\end{aligned}$$

and since the function $v(x)$ is zero on the boundary $\partial U$, one can choose $\delta_3>0$ such that for $\rho$ sufficiently small, all global minima of $\Lambda^+_{\psi_\infty,h_\infty}$ are contained in the set

$$U_{\delta_3}=\{x\in U\mid \operatorname{dist}(x,\partial U)\ge\delta_3\}.$$

Indeed, the definition of $\Lambda^+$ implies:

$$\Lambda^+_{\psi_\infty,h_\infty}(x)\ge\Lambda^+_{\phi_\infty,g_\infty}(x)-\rho(1+(v(x))^{-1}).$$

Further, (1.5.10), (1.5.11) yield the following inequalities:

$$\begin{aligned}\Lambda^+_{\psi_\infty,h_\infty}(x)&\ge\Lambda^+_{\phi_\infty,g_\infty}(x)-\rho(1+v(x)^{-1})\\&\ge\Lambda^+_{\phi_\infty,g_\infty}(\xi)+\delta_2-\rho(1+(v(x))^{-1})\\&\ge\Lambda^+_{\psi_\infty,h_\infty}(\xi)+\delta_2-\rho(2+(v(x))^{-1}+(v(\xi))^{-1})\\&\ge\min\Lambda^+_{\psi_\infty,h_\infty}+\frac{\delta_2}{2},\qquad\forall x\in U_{\delta_3}\setminus B_{\delta_1},\end{aligned}$$

provided that $\rho$ is sufficiently small. Therefore, for such $\rho$, all global minima of $\Lambda^+_{\psi_\infty,h_\infty}$ are contained in the set $B_{\delta_1}$. The following estimate, however, shows that the functions $\Lambda^+_{\psi_\infty,h_\infty}$ are strictly convex in $B_{\delta_1}$ for $\rho$ sufficiently small. Using the interior Schauder

estimate and (1.5.9), one finds for $x \in B_{\delta_1}$:

$$D_x^2 \Lambda^+_{\psi_\infty, h_\infty}(x) = D_x^2 \Lambda^+_{\phi_\infty, g_\infty}(x) +$$
$$+ D_x^2((v(x))^{-1}(\int_{\partial U} E(x,y')(\psi_\infty(y')-\phi_\infty(y'))d\sigma_{y'} +$$
$$+ \int_U G(x,y)(h_\infty(y)-g_\infty(y))dy$$

(1.5.12.) $D_x^2 \Lambda^+_{\psi_\infty, h_\infty}(x) \geq (\gamma - C\rho)\mathrm{Id}, \quad \forall x \in B_{\delta_1}$

where $C > 0$ is some constant.

Hence, $\Lambda^+_{\psi_\infty, h_\infty}$ has for $\rho << 1$ only one global minimum and the set $E_C(\psi_\infty, h_\infty)$ contains only one point $\eta = \eta(\psi_\infty, h_\infty) \in B_{\delta_1}$. (1.5.8) and (1.5.12) yield that the matrix of second derivatives of the corresponding critical solution at the point $\eta$ is positive definite. □

The next result follows immediately from the Theorems 1.5.1, 1.5.3:

Corollary 1.5.4. The complement of $O_+^\alpha$ in $C_+^0(\partial U) \times C^\alpha(\bar{U})$ is nowhere dense.

Theorem 1.5.5. Let $(\phi_\infty^0, g_\infty^0) \in O_+^\alpha$. Then the functional

(1.5.13.) $C_+^0(\partial U) \times C^\alpha(\bar{U}) \ni (\phi_\infty, g_\infty) \to \lambda_C^+(\phi_\infty, g_\infty) \in \mathbb{R}$

is Frechet-differentiable at $(\phi_\infty^0, g_\infty^0)$ and its first variation in the direction $(\psi_\infty, h_\infty)$ is given by the formula:

(1.5.14.) $\delta\lambda_C^+(\phi_\infty^0, g_\infty^0) \circ (\psi_\infty, h_\infty) = (v(\xi))^{-1}(\int_{\partial U} E(\xi, y')\psi_\infty(y')d\sigma_{y'} +$

$$+ \int_U G(\xi,y)h_\infty(y)dy).$$

Here $\{\xi\} = E_C(\phi_\infty^0, g_\infty^0)$, G is Green's function of the Dirichlet problem with zero boundary condition for the Poisson equation in U, and the

functions E and v are defined by $E(x,y') = \pi_0 \frac{\partial}{\partial n_{y'}} G(x,y')$ and (1.4.1), respectively.

Proof. First, one has to show that the function

$$(1.5.15.)\quad s \to \lambda_c^+(\phi_\infty^0+s\psi_\infty, g_\infty^0+sh_\infty)$$

is differentiable at $s = 0$, $\forall(\psi_\infty,h_\infty) \in C^0(\partial U) \times C^\alpha(\bar{U})$.

Since $O_+^\alpha$ is open in $C^0(\partial U) \times C^\alpha(\bar{U})$, one has the following formula:

$$(1.5.16.)\quad \lambda_c^+(\phi_\infty^0+s\psi_\infty, g_\infty^0+sh_\infty) = \Lambda^+_{\phi_\infty^0+s\psi_\infty,\, g_\infty^0+sh_\infty}(\xi(s))$$

where $\Lambda^+$ is defined by (1.3.3) and $\xi(s)$ is well defined by $\{\xi(s)\} = E_c(\phi_\infty^0+s\psi_\infty, g_\infty^0+sh_\infty)$ for $|s|$ sufficiently small.

Besides, one has for $\xi(s)$ the following equations:

$$\nabla_x \Lambda^+_{\phi_\infty^0+s\psi_\infty,\, g_\infty^0+sh_\infty}(\xi(s)) = 0.$$

Since, as a consequence of (1.5.8)),

$$D_x^2 \Lambda^+_{\phi_\infty^0,\, g_\infty^0}(\xi(0)) = v(\xi(0))^{-1} D_x^2 v_\infty^{\lambda_c}(\xi(0)) \geq \gamma \text{ Id},\ \gamma > 0,$$

the implicit function theorem yields: the function $s \to \xi(s) \in U$ is differentiable for $|s|$ sufficiently small. As a consequence of (1.5.16), the regularity of the function $\Lambda^+(x)$, $x \in U$ and the differentiability of $\xi(s)$, one gets the conclusion that the function (1.5.15) is differentiable at $s = 0$.

A straightforward computation using the relations:

$$v_\infty^{\lambda_c^+(\phi_\infty^0, g_\infty^0)}(\xi(0)) = 0, \quad \nabla_x v_\infty^{\lambda_c^+(\phi_\infty^0, g_\infty^0)}(\xi(0)) = 0,$$

then yields the formula (1.5.14). $\square$

Theorem 1.5.6. Let $U \subset \mathbb{R}^2$ be a bounded, simply connected domain and

let $\phi_\infty \in C^0_+(\partial U)$ , $g_\infty \equiv 0$ . If the coefficients $a_{kj}(x)$ of the differential operator $A_\infty\left(x,\frac{\partial}{\partial x}\right)$ are real analytic in $U$ , then the critical set $E_c(\phi_\infty,0)$ consists only of a finite number of points.

Theorem 1.5.7. If $U \subset \mathbb{R}^2$ is a bounded domain (not necessarily simply connected) and if the coefficients of the differential operator $A_\infty$ are real analytic, then $E_c(\phi_\infty,0)$ is the collection of finitely many isolated points and a finite number of closed analytic curves.

The proof of the Theorems 1.5.6, 1.5.7 stated above is similar to the proof of Theorem 5 and Corollary 2 in [9].

It should be mentioned that even if $U$ is the unit disk in $\mathbb{R}^2$ , there exist functions $\phi_\infty \in C^0_+(\partial U)$ , such that the set $E_c(\phi_\infty,0)$ contains more than one point. Moreover, in this case the set $E_0(u^\lambda_\infty)$ is not necessarily connected (see [9]).

## 1.6. The asymptotic behaviour of the solution to $\mathfrak{A}^\lambda_\infty$ when $\lambda \to +\infty$

In this section, an asymptotic formula for the solution $u^\lambda_\infty$ of $\mathfrak{A}^\lambda_\infty$ is indicated and an error estimate in the maximum norm is stated. One uses super- and subsolutions of special type and the maximum principle for establishing this result (see [9], Theorem 7 for the case $A_\infty = -\Delta$ ).

First, consider the case

$$f_{-\infty} < 0 < f_\infty . \tag{1.6.1}$$

For simplicity, it will be assumed from now on, that

$$\phi_\infty \in C^0_+ \cap C^2(\partial U) \tag{1.6.2}$$

For $x \in U$ in a sufficiently small neighbourhood of $\partial U$ , let $x' \in \partial U$ and $\rho$ be defined by

$$|x-x'| = \mathrm{dist}(x,\partial U) = \min_{y'\in\partial U} |x-y'| = \rho .$$

Using the coordinates $x',\rho$ , one can rewrite the operator $A_\infty$ as follows:

$$A_\infty\left(x,\frac{\partial}{\partial x}\right) = -a(x',\rho)\frac{\partial^2}{\partial\rho^2} + B$$

Here $a(x',\rho)$ is a smooth positive function and the differential operator $B$ has orders 2 and 1 in $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial \rho}$, respectively. Using the notation

$$a(x') = a(x',0) = \sum_{1\le k,j\le n} a^{\infty}_{kj}(x')N_k(x')N_j(x')$$

and the normalized distance $\rho_1$,

$$\rho_1 = \left(\lambda f_\infty \cdot (2a(x'))^{-1}\right)^{1/2} \rho \quad ,$$

one has:

<u>Theorem 1.6.1.</u> Under the assumptions (1.6.1), (1.6.2), the function

$$w^\lambda_\infty(x) = \left((\phi_\infty(x'))^{1/2} - \rho_1\right)^2_+ , \tag{1.6.3}$$

where $s_+ = \max(s,0)$, is an asymptotic solution of $\mathfrak{A}^\lambda_\infty$ such that

$$\left\| u^\lambda_\infty - w^\lambda_\infty \right\|_{C^0(\overline{U})} \le C\lambda^{-1/2} , \tag{1.6.4}$$

where the constant $C$ does not depend on $\lambda$. Moreover, for the free boundary $\partial E_0(u^\lambda_\infty)$ holds:

$$\partial E_0(u^\lambda_\infty) \subset \left\{ x \in U \;\middle|\; \left| \left(2\phi_\infty(x')a(x')(\lambda f_\infty)^{-1}\right)^{1/2} - \rho \right| \le C_1\lambda^{-1} \right\} \tag{1.6.5}$$

where the constant $C_1$ does not depend on $\lambda$.

Assuming again (1.6.2), we consider now the case $0 = f_{-\infty} < f_\infty$.

If $g_\infty(x) \ge 0 \;\; \forall x \in \overline{U}$, then the function (1.6.3) is still an asymptotic solution of $\mathfrak{A}^\lambda_\infty$ when $\lambda \to \infty$. Using super- and subsolutions, one shows that (1.6.4), (1.6.5) are valid in this case, as well.

If $g_\infty(x) < 0 \;\; \forall x \in \overline{U}$, $g_\infty \in C^\alpha(\overline{U})$, then it will turn out that for $\lambda \to \infty$, the solution $u^\lambda$ converges on any $U_1 \subset\subset U$ to the solution $z(x)$ of the linear boundary value problem

$$A_\infty\left(x,\frac{\partial}{\partial x}\right)z(x) = g_\infty(x) , \quad x \in U$$
$$\pi_0 z(x') = 0 , \quad x' \in \partial U .$$

In a neighbourhood of $\partial U$ the solution $u_\infty^\lambda$ has again a boundary layer behaviour as $\lambda \to \infty$.

Let $\beta(x') = \pi_0 \frac{\partial z}{\partial N}(x')$ , $x' \in \partial U$ , and define $\xi(x')$ to be the positive solution of the quadratic equation $\beta(x')\xi(x') + \left(\lambda f_\infty / 2a(x')\right) \xi(x')^2 = \phi_\infty(x')$ . Let $U_\xi = \left\{ x \in U \mid \rho < \xi(x') \right\}$ and denote by $w_\infty^\lambda(x)$ , $x \in U \setminus U_\xi$ , the solution of the following boundary value problem:

$$
\text{(1.6.6)} \qquad \begin{aligned} &A_\infty\left(x, \frac{\partial}{\partial x}\right) w_\infty^\lambda(x) = g_\infty(x) , && x \in U \setminus U_\xi \\ &w_\infty^\lambda(x') = 0 , && x' \in \partial U_\xi \setminus \partial U \end{aligned}
$$

On the set $U_\xi$ , we define $w_\infty^\lambda$ as follows:

$$
w_\infty^\lambda(\rho, x') = \beta_\lambda(x')\left(\xi(x') - \rho\right) + \left(\lambda f_\infty / 2a(x')\right)\left(\xi(x') - \rho\right)^2 ,
$$

$$
0 < \rho < \xi(x')
$$

where $\beta_\lambda(x')$ denotes the restriction of the normal derivative to $\partial U_\xi / \partial U$ of the solution of (1.6.6).

The function $w_\infty^\lambda$ defined above, is a formal asymptotic solution of $\mathcal{O}\!\mathcal{L}_\infty^\lambda$ when $\lambda \to \infty$ :

$$
\begin{aligned} &A_\infty\left(x, \frac{\partial}{\partial x}\right) w_\infty^\lambda + \left(\lambda + O(\sqrt{\lambda})\right) f_0(w_\infty^\lambda) + g_\infty(x) \chi_0(w_\infty^\lambda) = g_\infty(x) , && x \in U \\ &0 \le g_\infty(x) \le \lambda f_\infty , && x \in \operatorname{int} E_0(w_\infty^\lambda) \\ &\pi_0 w_\infty^\lambda(x') = \phi_\infty(x') + O(\lambda^{-1/2}) , && x' \in \partial U \end{aligned}
$$

Note that the boundary condition is satisfied asymptotically because $\xi(x') = O(\lambda^{-1/2})$ when $\lambda \to \infty$ holds uniformly w.r.t. $x' \in \partial U$ and because the Schauder estimate implies that $[\beta - \beta_\lambda]_{C^0(\partial U)} = O(\lambda^{-1/2})$ when $\lambda \to \infty$. Using a maximum principle argument, one finds that $w_\infty^\lambda$ is an asymptotic solution of $\mathcal{O}\!\mathcal{L}_\infty^\lambda$ when $\lambda \to \infty$.

Note that in the case $f_{-\infty} = 0 < f_\infty$ , independently on the sign of $g(x)$ , $u^\lambda$ converges to the solution $u$ of the following problem $\mathcal{O}\!\mathcal{L}_\infty$ :

$$
\begin{aligned} &A_\infty\left(x, \frac{\partial}{\partial x}\right) u = g_\infty(x)\left(1 - \chi_0(u)\right) , && x \in U \\ &0 \le g_\infty(x) , && x \in \operatorname{int}(E_0(u)) \\ &u(x) \le 0 , && x \in U \\ &\pi_0 u(x') = 0 , && x' \in \partial U \end{aligned}
$$

In general, $\mathfrak{A}_\infty$ is, of course, a free boundary problem.

It can be reformulated as a minimization problem as follows:

$$\min_{\substack{u \in \overset{\circ}{H}_1(U) \\ u \le 0}} \int_U \left( \frac{1}{2} \sum a_{kj}(x) u_{x_j} u_{x_k} - g_\infty(x) u(x) \right) dx$$

Finally, consider the case $f_{-\infty} < 0 = f_\infty$.

For $\lambda \geq \max_{x \in U} \left( f_{-\infty}^{-1} g_\infty(x) \right)$, the function $u_\infty^\lambda$ does not depend upon $\lambda$ and coincides with the solution $u$ of the following problem $\mathfrak{A}_\infty$:

$$\begin{aligned} A_\infty u(x) &= g_\infty(x)\left(1 - \chi_0(u)\right), & x &\in U \\ g_\infty(x) &\le 0, & x &\in \operatorname{int}(E_0(u)) \\ u(x) &\ge 0, & x &\in U \\ \pi_0 u(x') &= \phi_\infty(x'), & x' &\in \partial U. \end{aligned}$$

The equivalent formulation as a minimization problem reads as follows:

$$\min_{\substack{u \in H_1(U) \\ \pi_0 u = \phi_\infty \\ u \ge 0}} \int_U \left( \frac{1}{2} \sum a_{kj}(x) u_{x_k} u_{x_j} - g_\infty(x) u(x) \right) dx$$

<u>Remark 1.6.2</u>. One can consider a problem with more general non-linearity:

$$(1.6.7) \quad \begin{aligned} -\Delta u_\infty^\lambda + \lambda q'(u^\lambda) \chi_+(u_\infty^\lambda) &= 0, & x &\in U \\ \pi_0 u_\infty^\lambda(x') &= \phi(x'), & x' &\in U \end{aligned}$$

where $\phi(x') > 0$, $\forall x' \in \partial U$, $\lambda >> \phi$ and $q'(s)$ is monotonically increasing on the interval $[0, \phi_M]$ with $\phi_M = \max_{x' \in \partial U} \phi(x')$.

Then the corresponding asymptotic solution $w_\infty^\lambda(x', \rho)$ is defined by the formula:

$$(1.6.8) \quad \int_{w_\infty^\lambda(x',\rho)}^{\phi(x')} \frac{ds}{\sqrt{2q(s)}} = \sqrt{\lambda}\, \rho,$$

where $q(s)$ is the primitive of $q'(s)$ normalized by the condition $q(0) = 0$.

Moreover for the free boundary $\partial E_0(u_\infty^\lambda)$ of $u_\infty^\lambda$, solution to (1.6.7), holds:

$$(1.6.9) \qquad \partial E_0(u\ ) \subset \{x \in U \mid \left| \int_0^{\phi(x')} \frac{ds}{\sqrt{2\rho q(s)}} - \rho \right| \le C_1 \lambda^{-1}\},$$

where $C_1 > 0$ is some constant.

In case of a general second order elliptic operator the distance $\rho$ in (1.6.8), (1.6.9) has to be replaced by the normalized distance $\rho_1$ defined here above.

## II. Non-stationary problem

### 2.1. General properties of the operators considered

Denote

$$(2.1.1) \qquad B(Q_T) = L^2(Q_T) \times H_1(U) \times H_{3/2,3/4}(\Gamma_T) , \quad 0 < T < \infty ,$$

where $H_{s,r}(\Gamma_T)$ is the Sobolev space of all functions $\phi(x',t)$ such that $D_x^\alpha, D_t^m \phi \in L^2(\Gamma_T)$ $\forall |\alpha| \le s$ , $m \le r$ for $s \ge 0$ , $r \ge 0$ integer; if $s$ and $r$ are not necessarily integer, then $H_{s,r}(\Gamma_T)$ is defined in a standard way by using the partition of unity and the Fourier transform. Denote by $\mathfrak{A}_\varepsilon^\lambda$ the operator associated with the initial-boundary value problem (0.6):

$$(2.1.2) \qquad \mathfrak{A}_\varepsilon^\lambda : H_{2,1}(Q_T) \to B(Q_T) , \quad 0 < T < \infty$$

Theorem 2.1.1. For any given $\varepsilon > 0$ , the mapping (2.1.2) is a Lipschitz-continuous homeomorphism.

Theorem 2.1.2. If $(g,\psi,\phi) \in C^0(\bar{Q}) \times C^2(\bar{U}) \times C^{2,1}(\Gamma)$ and the compatibility condition (0.7) is satisfied, then for $\forall \alpha \in [0,1)$ uniformly with respect to $\varepsilon \in (0,\varepsilon_0]$ holds:

$$u_\varepsilon^\lambda \in C^{1,\alpha;(1+\alpha)/2}(\bar{Q}) .$$

Theorem 2.1.3. If $(g,\psi,\phi) \in C^0(\bar{Q}) \times C^2(\bar{U}) \times C^{2,1}(\Gamma)$ and (0.7) is satisfied, then the reduced problem $\mathfrak{A}^\lambda$ has a well-defined (distributional) solution $u^\lambda \in C^{1,\alpha;(1+\alpha)/2}(\bar{Q})$ $\forall \alpha \in [0,1)$ .

Moreover, the set $\{u_\varepsilon^\lambda\}_{0<\varepsilon\le\varepsilon_0} \subset C^{1,\alpha;(1+\alpha)/2}(Q_T)$ , where $u_\varepsilon^\lambda$ is the solution of $\mathfrak{A}_\varepsilon^\lambda$ , has for $\forall T < \infty$ as its only condensation point the solution $u^\lambda$ of $\mathfrak{A}^\lambda$ when $\varepsilon \downarrow 0$ , so that

$$(2.1.3) \qquad \lim_{\varepsilon \downarrow 0} \left\| u_\varepsilon^\lambda - u^\lambda \right\|_{C^{1,\alpha;(1+\alpha)/2}(\bar{Q}_T)} = 0 \quad \forall T < \infty , \quad \forall \alpha \in [0,1) .$$

The proof of the Theorems 2.1.1, 2.1.2 and 2.1.3 will be given in a coming authors' publication.

## 2.2. Convergence as $t \to \infty$

For simplicity, we assume that

$$(2.2.1) \qquad A\left(x,t,\frac{\partial}{\partial x}\right) \equiv A_\infty\left(x,\frac{\partial}{\partial x}\right),\ \varphi(x',t) \equiv \phi_\infty(x')\ ,\ g(x,t) \equiv g_\infty(x)\ .$$

Let $\mu > 0$ be the least eigenvalue of $A_\infty$ in $U$ with Dirichlet boundary condition on $\partial U$.

Theorem 2.2.1. The stationary solutions $u^\lambda_{\varepsilon,\infty}$, $u^\lambda_\infty$ of the problems $\mathfrak{A}^\lambda_\varepsilon$ and $\mathfrak{A}^\lambda$, respectively, are asymptotically stable in $L^2(U)$ as $t \to \infty$, and, moreover, the following estimates hold:

$$(2.2.2) \quad \begin{cases} \left[u^\lambda_\varepsilon(\cdot,t) - u^\lambda_{\varepsilon,\infty}\right]_{L^2(U)} \le e^{-\mu t}\left[\psi - u^\lambda_{\varepsilon,\infty}\right]_{L^2(U)} & \forall t \ge 0\ , \quad \forall \varepsilon > 0 \\ \left[u^\lambda(\cdot,t) - u^\lambda_\infty\right]_{L^2(U)} \le e^{-\mu t}\left[\psi - u^\lambda_\infty\right]_{L^2(U)} & \forall t \ge 0\ . \end{cases}$$

Proof. The difference $w^\lambda_\varepsilon = u^\lambda_\varepsilon - u^\lambda_{\varepsilon,\infty}$ is the solution of the following problem:

$$\frac{\partial w^\lambda_\varepsilon}{\partial t} + A_\infty\left(x,\frac{\partial}{\partial x}\right)w^\lambda_\varepsilon + \lambda\left(f(\varepsilon^{-1}u^\lambda_\varepsilon) - f(\varepsilon^{-1}u^\lambda_{\varepsilon,\infty})\right) = 0\ ,\quad (x,t) \in Q$$

$$w^\lambda_\varepsilon(x,0) = \psi(x) - u^\lambda_{\varepsilon,\infty}(x)\ ,\quad x \in \overline{U}$$

$$\pi_0 w^\lambda_\varepsilon(x',t) = 0\ ,\quad (x',t) \in \Gamma$$

Multiplying the differential equation with $w^\lambda_\varepsilon$, integrating by parts and using the fact that $f$ is monotonically increasing, one gets the following inequality:

$$(2.2.3) \qquad \frac{1}{2}\frac{d}{dt}\left[w^\lambda_\varepsilon(\cdot,t)\right]^2_{L^2(U)} + (A_\infty w^\lambda_\varepsilon, w^\lambda_\varepsilon) \le 0\ ,\quad \forall t \ge 0$$

Now the inequality

$$(2.2.4) \qquad (A_\infty w,w) \ge \mu[w]^2_{L^2(U)} \quad \forall w \in \overset{0}{H}_1(U)$$

and Gronwall's Lemma yield the first of the inequalities (2.2.2). The second inequality in (2.2.2) follows from the first one and from (2.1.3). □

**Theorem 2.2.2.** The following estimates hold under the assumption (2.2.1):

$$\text{(2.2.5)} \qquad \left[(u_\varepsilon^\lambda)_t(\cdot,t)\right]_{L^2(U)} \le e^{-\mu t}\left[A_\infty\psi + \lambda f(\varepsilon^{-1}\psi) - g_\infty\right]_{L^2(U)} \quad \forall t,\varepsilon > 0 .$$

$$\text{(2.2.6)} \qquad \left(A_\infty(u_\varepsilon^\lambda - u_{\varepsilon,\infty}^\lambda), u_\varepsilon^\lambda - u_{\varepsilon,\infty}^\lambda\right)^{1/2}_{L^2(U)} \le e^{-\mu t}\left[A_\infty\psi + \lambda f(\varepsilon^{-1}\psi) - g_\infty\right]^{1/2}_{L^2(U)} \cdot$$
$$\cdot \left[\psi - u_{\varepsilon,\infty}^\lambda\right]^{1/2}_{L^2(U)} \quad \forall t,\varepsilon > 0$$

*Proof.* Denote $v_\varepsilon^\lambda = (u_\varepsilon^\lambda)_t$ , so that $v_\varepsilon^\lambda$ is the solution of the following problem:

$$\text{(2.2.7)} \qquad \begin{cases} \dfrac{\partial v_\varepsilon^\lambda}{\partial t} + A_\infty\left(x,\dfrac{\partial}{\partial x}\right)v_\varepsilon^\lambda + \varepsilon^{-1}\lambda f'(\varepsilon^{-1}u_\varepsilon^\lambda)v_\varepsilon^\lambda = 0 , & (x,t) \in Q \\ v_\varepsilon^\lambda(x,0) = g_\infty - A_\infty\psi - \lambda f(\varepsilon^{-1}\psi) , & x \in \overline{U} \\ \tau_0 v_\varepsilon^\lambda(x',t) = 0, & (x',t) \in \Gamma \end{cases}$$

Since $f'(s) \ge 0$ , one gets, using (2.2.7):

$$\text{(2.2.8)} \qquad \frac{1}{2}\frac{d}{dt}\left[v_\varepsilon^\lambda(\cdot,t)\right]^2_{L^2(U)} + (A_\infty v_\varepsilon^\lambda, v_\varepsilon^\lambda) \le 0 \quad \forall t \ge 0 .$$

Further, (2.2.4) and Gronwall's lemma yield (2.2.5). Using (2.2.3), (2.2.4) and (2.2.5), one obtains:

$$(A_\infty w_\varepsilon^\lambda, w_\varepsilon^\lambda)^{1/2}_{L^2(U)} \le \left[(u_\varepsilon^\lambda)_t(\cdot,t)\right]^{1/2}_{L^2(U)} [w_\varepsilon^\lambda]^{1/2}_{L^2(U)}$$
$$\le e^{-\mu t}\left[A_\infty\psi + \lambda f(\varepsilon^{-1}\psi) - g_\infty\right]^{1/2}_{L^2(U)}\left[\psi - u_{\varepsilon,\infty}^\lambda\right]^{1/2}_{L^2(U)} .$$

□

## 2.3. Convergence for $\varepsilon \downarrow 0$ .

For simplicity, it is assumed that

$$\text{(2.3.1)} \qquad g \equiv 0 , \ \psi \ge 0 , \ \phi \ge 0 .$$

**Proposition 2.3.1.** Under the assumptions (0.7), (2.3.1), the following estimates hold:

$$\text{(2.3.2)} \qquad \left\| u_\varepsilon^\lambda(x,t) - u^\lambda(x,t) \right\|_{L^2([0,T],H_1(U))} \le C(T\varepsilon)^{1/2} \quad \forall T > 0 ,$$
$$\left\| u_\varepsilon^\lambda(x,t) - u^\lambda(x.t) \right\|_{L^2([0,\infty],L_2(U))} \le C\varepsilon^{1/2},$$

where the constant $C$ depends only upon $\lambda$, meas $U$, $f$, the ellipticity constant of $A$ and upon the first eigenvalue of $(-\Delta)$ in $U$ with Dirichlet boundary conditions on $\partial U$.

<u>Proof.</u> The difference $v_\varepsilon^\lambda = u_\varepsilon^\lambda - u^\lambda$ is a solution of the following problem.

$$(2.3.3)\quad \begin{cases} (v_\varepsilon^\lambda)_t - A\, v_\varepsilon^\lambda + \lambda\left(f\left(\frac{u_\varepsilon^\lambda}{\varepsilon}\right) - f_0(u^\lambda)\right) = 0\,, & (x,t) \in Q \\ v_\varepsilon^\lambda(x,0) = 0\,, & x \in U \\ \pi_0 v_\varepsilon^\lambda(x',t) = 0\,, & x' \in \partial U\,,\ t \in \mathbb{R}_+ \end{cases}$$

Multiplying the differential equation with $v_\varepsilon^\lambda$ and integrating by parts, one obtains

$$\frac{1}{2}\frac{d}{dt}[v_\varepsilon^\lambda]^2_{L^2(U)} + \sum_{k,j}\int_U a_{kj}(x,t)(v_\varepsilon^\lambda)_k(v_\varepsilon^\lambda)_j dx + \lambda \int_{E_0(u^\lambda)\cup E_-(u^\lambda)} f\left(\frac{u_\varepsilon^\lambda}{\varepsilon}\right)v_\varepsilon^\lambda dx =$$

$$= \lambda \int_{E_+(u^\lambda)} \left(f_0(u^\lambda) - f\left(\frac{u_\varepsilon^\lambda}{\varepsilon}\right)\right)v_\varepsilon^\lambda dx$$

As a consequence of the assumption (2.3.1) and of Proposition 2.4.1 below, $E_-(u^\lambda) = \emptyset$ and $v_\varepsilon^\lambda \geq 0$. Thus,

$$\frac{1}{2}\left[v_\varepsilon^\lambda(\cdot,T)\right]^2_{L^2(U)} + \gamma\int_{Q_T} |\nabla_x v_\varepsilon^\lambda|^2 dxdt \leq \lambda \int_{Q_T}\left(f_\infty - f\left(\frac{u_\varepsilon^\lambda}{\varepsilon}\right)\right)v_\varepsilon^\lambda\, dxdt$$

where $\gamma > 0$ is the ellipticity constant of $A$.

Thus,

$$(2.3.4)\qquad \frac{1}{2}\frac{d}{dt}[v_\varepsilon^\lambda]^2_{L^2(U)} + \gamma\mu[v_\varepsilon^\lambda]^2_{L^2(U)} \leq \lambda \int_{E_+(u^\lambda)} (f_0(u^\lambda) - f(\frac{u_\varepsilon^\lambda}{\varepsilon}))v_\varepsilon^\lambda\, dx$$

where $\gamma > 0$ si the ellipticity constant and $\mu > 0$ is the least eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition on $\partial U$.

The integral in the right hand side of the last inequality can be estimated as follows:

$$(2.3.5)\qquad \begin{aligned} &\int_U (f_\infty - f(\frac{u_\varepsilon^\lambda}{\varepsilon}))v_\varepsilon^\lambda\, dx \leq C\int_U \frac{\varepsilon}{\varepsilon+u_\varepsilon^\lambda}(u_\varepsilon^\lambda - u^\lambda)dx \leq \\ &\leq C\varepsilon\int_U \frac{u_\varepsilon^\lambda}{\varepsilon+u_\varepsilon^\lambda}\, dx \leq C\ \varepsilon \text{ meas } U. \end{aligned}$$

Using (2.3.4), (2.3.5) and Gronwall's lemma, one gets the second of the inequalities (2.3.3).

Integrating (2.3.4) over $[0,T]$ one gets the first of the inequalities (2.3.2).

Similarly to section 1.2, an improved estimate for the rate of convergence of $u_\varepsilon^\lambda$ to $u^\lambda$ can be obtained under the same assumptions upon the free boundary of the solution $u^\lambda$. However, the construction of asymptotic solutions in this case is somewhat more tedious than for the stationary problem.

### 2.4. Nonnegative solutions

Proposition 2.4.1. If $g \geq 0$, $\psi \geq 0$, $\phi \geq 0$, then $u_\varepsilon^\lambda \geq 0$, $u^\lambda \geq 0$.

Proof. Assume that the set $E_-(u_\varepsilon^\lambda)$ is nonempty. For $\forall (x,t) \in E_-(u_\varepsilon^\lambda)$, one has:

$$(u_\varepsilon^\lambda)_t + A\, u_\varepsilon^\lambda = g(x,t) - \lambda f(\varepsilon^{-1} u_\varepsilon^\lambda) \geq 0 .$$

Moreover, $u_\varepsilon^\lambda(x,t) = 0$ for $\forall (x,t) \in \partial E_-(u_\varepsilon^\lambda)$. The maximum principle yields $u_\varepsilon^\lambda \geq 0$ in $E_-(u_\varepsilon^\lambda)$. Thus one obtains a contradiction. The nonnegativity of $u^\lambda$ is proved similarly.

□

Let $v_\varepsilon^\lambda$ be the solution of the following linear problem:

$$(2.4.1)\qquad \begin{cases} \dfrac{\partial v_\varepsilon^\lambda}{\partial t} + A\left(x,t,\dfrac{\partial}{\partial x}\right) v_\varepsilon^\lambda + \dfrac{\lambda}{\varepsilon} f'(0)\, v_\varepsilon^\lambda = g\,, & (x,t) \in Q \\ v_\varepsilon^\lambda(x,0) = \psi(x)\,, & x \in \overline{U} \\ \pi_0 v_\varepsilon^\lambda(x',t) = \varphi(x',t)\,, & (x',t) \in \Gamma\,. \end{cases}$$

Proposition 2.4.2. Assume $f(s)$ to be concave and the data to be nonnegative. Then $u_\varepsilon^\lambda(x,t) \geq v_\varepsilon^\lambda(x,t) \forall (x,t) \in \overline{Q}$, where $v_\varepsilon^\lambda$ is the solution of (2.4.1).

Proof. The difference $w_\varepsilon^\lambda = u_\varepsilon^\lambda - v_\varepsilon^\lambda$ is a solution of the following problem:

$$\begin{aligned} &(w_\varepsilon^\lambda)_t + A w_\varepsilon^\lambda + \lambda\left(f(\varepsilon^{-1}u_\varepsilon^\lambda) - f\left(\varepsilon^{-1}(u_\varepsilon^\lambda - w_\varepsilon^\lambda)\right)\right) = h_\varepsilon(x,t)\,, && (x,t) \in Q \\ &w_\varepsilon^\lambda(x,0) = 0\,, && x \in U \\ &\pi_0 w_\varepsilon^\lambda(x',t) = 0\,, && (x',t) \in \Gamma\,, \end{aligned}$$

where the function

$$h_\varepsilon = -f(\varepsilon^{-1}v_\varepsilon^\lambda) + \varepsilon^{-1}f'(0)v_\varepsilon^\lambda$$

is nonnegative since $f$ is concave. The maximum principle yields: $w_\varepsilon^\lambda \geq 0$ in $\overline{Q}$.

□

<u>Proposition 2.4.3.</u> If $u_\varepsilon^\lambda \geq 0$, then $u_\varepsilon^\lambda$ is a monotonically increasing function of $\varepsilon > 0$.

<u>Proof.</u> Let $w_\varepsilon^\lambda = \dfrac{\partial u_\varepsilon^\lambda}{\partial \varepsilon}$. Then $w_\varepsilon^\lambda$ is the solution of the problem

$$\begin{cases} \dfrac{\partial w_\varepsilon^\lambda}{\partial t} + Aw_\varepsilon^\lambda + \varepsilon^{-1}\lambda f'(\varepsilon^{-1}u_\varepsilon^\lambda)w_\varepsilon^\lambda = \varepsilon^{-2}\lambda f'(\varepsilon^{-1}u_\varepsilon^\lambda)u_\varepsilon^\lambda, & (x,t) \in Q, \\ w_\varepsilon^\lambda(x,0) = 0, & x \in \overline{U} \\ \tau_0 w_\varepsilon^\lambda(x',t) = 0, & (x',t) \in \Gamma \end{cases}$$

As a consequence of the maximum principle, one finds $w_\varepsilon^\lambda(x,t) \geq 0$, $\forall (x,t) \in \overline{Q}$, since $f'(s) \geq 0$, $u_\varepsilon^\lambda \geq 0$.

□

## 2.5. <u>Special solutions of Cauchy's problem for the reduced operator</u>

In this section, we assume that

$$\begin{aligned} &A\left(x,t,\frac{\partial}{\partial x}\right) = -\Delta \\ &U = \mathbb{R}^n \\ &f_0(s) = \operatorname{sgn} s, \ \forall s \in \mathbb{R}\setminus\{0\}, \ f_0(0) = 0 \\ &g(x,t) = 0 \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ . \end{aligned}$$

We indicate the following two types of special nonnegative solutions of Cauchy's problem for the reduced operator.

(i) Travelling waves' solutions:

$$u^\lambda(x,t;w,\xi) = \lambda w^{-1}(x\cdot\xi - wt)_+ - \lambda|\xi|^2 w^{-2}\left[1 - \exp\left(-w|\xi|^{-2}(x\cdot\xi - wt)_+\right)\right] \tag{2.5.1}$$

where $\xi \in \mathbb{R}^n$, $w \in \mathbb{R}\setminus\{0\}$ and $s_+ = \max(s,0)$.

(ii) Similarity solutions:

These are solutions of the form

$$u^{\lambda}(x,t) = t\, v\left(|x|\, t^{-1/2}\right), \tag{2.5.2}$$

where $v(s)$, $s \in \mathbb{R}_+$, are nonnegative solutions of the following ordinary differential equation

$$-v''(s) - \left(\frac{s}{2} + \frac{n-1}{s}\right)v'(s) + v(s) + \lambda\chi_+(v(s)) = 0, \quad s > 0, \tag{2.5.3}$$

so that $v(s) \geq 0$ is given by the formula

$$v(s) = (s^2 + 2n)\left[c_1 \int_1^s \xi^{1-n}(\xi^2 + 2n)^{-2} e^{-\xi^2/4} d\xi + c_2\right] - \lambda \tag{2.5.4}$$

If $n = 1$, one finds the solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial u^{\lambda}}{\partial t} - \frac{\partial^2 u^{\lambda}}{\partial x^2} + \lambda\chi_+(u^{\lambda}) = 0, \quad x \in \mathbb{R}, \quad t > 0 \\ &u^{\lambda}(x,0) = x_+^2 \end{aligned} \tag{2.5.5}$$

which has the form (2.5.2), where

$$v(s) = \left\{(s^2 + 2)\left[a + b \int_0^s (\xi^2 + 2)^{-2} \exp(-\xi^2/4)\, d\xi\right] - \lambda\right\} H(s - \alpha). \tag{2.5.6}$$

In this case, the free boundary is a parabola $x = \alpha\sqrt{t}$. One gets a system of three equations for the parameters $a, b, \alpha$, which after elimination of $a$ and $b$ leads to the following functional equation for the free boundary parameter $\alpha$:

$$(\alpha^2 + 2)^{-1} - 2\alpha \exp(\alpha^2/4) \int_{\alpha}^{\infty} (\xi^2 + 2)^{-2} \exp(-\xi^2/4)\, d\xi = \lambda^{-1}. \tag{2.5.7}$$

For $\forall\lambda > 0$ the equation (2.5.7) has a well-defined solution $\alpha \in \mathbb{R}$.
For $\lambda = 2$, one gets the stationary solution $u(x,t) = x_+^2$.

## 2.6. Short time asymptotics

Consider the Cauchy problem:

$$\begin{cases} u_t^{\lambda} - u_{xx}^{\lambda} + \lambda\chi_+ (u^{\lambda}) = 0, & x \in \mathbb{R},\ t > 0 \\ u^{\lambda}(x,0) = \psi(x) \end{cases} \tag{2.6.1}$$

where $\psi(x) \equiv 0$ for $x < 0$, $\psi(x) = \gamma x_+^2 + O(x_+^3)$ for $x_+ \to 0$ and $\psi(x) \le Cx^2$, $\forall x \in \mathbb{R}_+$ with some constant $C > 0$.

Denote by $w^\lambda(x,t)$ the corresponding similarity solution:

$$(2.6.2) \quad \begin{cases} w_t^\lambda - w_{xx}^\lambda + \lambda \chi_+(w^\lambda) = 0, & x \in \mathbb{R},\ t > 0 \\ w^\lambda(x,0) = \gamma x_+^2, & \end{cases}$$

$w^\lambda(x,t)$ being defined by the formula $w^\lambda(x,t) = tv^\lambda\left(\frac{x}{\sqrt{t}}\right)$ with $v^\lambda(s)$ given by (2.5.6), (2.5.7).

Then the following estimate holds:

$$(2.6.3) \quad \sup_{(x,t)\in Q_{a,T}^\delta} |u^\lambda(x,t) - w^\lambda(x,t)| \le C_{a,T}\delta, \quad \forall \delta > 0,$$

where $Q_{a,T}^\delta = \{(x,t) \mid 0 \le t \le T\delta^2,\ |x| \le a\delta\}$ and the constant $C_{a,T}$ does not depend of $\delta$.

One proves (2.6.3) using the method introduced in [21].

## 2.7. Asymptotics for $\lambda \to \infty$

In order to avoid unnecessary technical complications we consider here the case of one ~~special~~ variable $x \in U = (-1,1)$ and of special (constant) space initial and boundary conditions. Namely, consider the problem $\mathcal{P}_\varepsilon^\lambda$:

$$(2.7.1) \quad \begin{cases} u_t^\lambda - u_{xx}^\lambda + \lambda\chi_+(u^\lambda) = 0, & (x,t) \in U \times \mathbb{R}_+ \\ u^\lambda(x,0) = 1, & x \in U \\ \pi_0 u^\lambda(x',t) = 1, & (x',t) \in \partial U \times \mathbb{R}_+ \end{cases}$$

Let $v(x,t)$ be the solution of the problem:

$$(2.7.2) \quad \begin{cases} v_t - v_{xx} = 1, & (x,t) \in U \times \mathbb{R}_+ \\ v(x,0) = 0, & x \in U \\ \pi_0 v(x',t) = 0, & (x',t) \in \partial U \times \mathbb{R}_+, \end{cases}$$

and let $t_c(\lambda)$ be defined by the relation:

$$(2.7.3) \quad v(0,t_c(\lambda)) = \lambda^{-1}.$$

Then for $t \in [0,t_c(\lambda))$ holds:

$$u^\lambda(x,t) = 1 - \lambda v(x,t)$$

Denote

$$(2.7.4) \quad \gamma = \gamma(\lambda) = \lambda(1 - v(0,t_c(\lambda))$$

and let $w_\gamma^\lambda(x,t)$ be the similarity solution of the Cauchy problem:

$$
(2.7.5)\quad \begin{cases} (w_\gamma^\lambda)_t - (w_\gamma^\lambda)_{xx} + \lambda\chi_+(w_\gamma^\lambda) = 0\,, & (x,t) \in \mathbb{R}\times\mathbb{R}_+ \\ w_\gamma^\lambda(x,0) = \gamma x^2 & \end{cases}
$$

Theorem 2.7.1. For any constants $T > 0$, $a > 0$, there exists a constant $C = C(T,a)$ such that for the solution $u^\lambda(x,t)$ the following inequality holds:

$$
(2.7.6)\quad \sup_{\substack{0\le t-t_c(\lambda)\le T\lambda^{-1} \\ |x|\le a\lambda^{-\frac{1}{2}}}} |u^\lambda(x,t)-w_\gamma^\lambda(x,t-t_c(\lambda))| \le C(T,a)\lambda^{-1},
$$

where $t_c(\lambda)$, $\gamma$, $w_\gamma^\lambda(x,t)$ are defined by (2.7.2)-(2.7.5).

$\lambda^{-1}+o(\lambda^{-1})$

Remark 2.7.2. One finds easily that $t_c(\lambda) = \boxed{\lambda^{-1}+o(\lambda^{-1})}$. Furthermore, for $0 \le t < t_c(\lambda)$ the following asymptotic formula holds:

$$
(2.7.7)\quad u^\lambda(x,t) = (1-\lambda t) + O(\lambda^{-1}) \text{ for } \lambda \to \infty
$$

while for $\sqrt{\lambda}t >> 1$ one has:

$$
(2.7.8)\quad u^\lambda(x,t) \sim u_\infty^\lambda(x), \qquad \lambda \to \infty,
$$

where

$$
u_\infty^\lambda(x) = (\lambda/2)(|x|-1+\sqrt{2\lambda^{-1}})_+^2
$$

is the solution of the corresponding stationary problem.

For $t_c(\lambda) < t < c\lambda^{-\frac{1}{2}}$ one has:

$$
(2.7.9)\quad u^\lambda(x,t) \sim v(\lambda(t-t_c(\lambda)),\ \sqrt{\lambda}\,\mathrm{dist}(x,\partial U)), \qquad \lambda \to \infty,
$$

where $v(\tau,\zeta)$ is the solution of the problem:

$$
(2.7.10)\quad \begin{cases} v_\tau - v_{\zeta\zeta} + \chi_+(v) = 0, & (\zeta,\tau) \in \mathbb{R}_+\times\mathbb{R}_+ \\ v(\zeta,0) = 1, & \zeta \in \mathbb{R}_+ \\ v(0,\tau) = 1, & \tau \in \mathbb{R}_+ \end{cases}
$$

Introducing $w = v_\tau$ one can reformulate (2.7.10) as the following slightly modified Stefan problem:

$$
(2.7.11)\begin{cases}
w_\tau - w_{\zeta\zeta} = 0, & 0 < \zeta < s(\tau),\ \tau \in \mathbb{R}_+ \\
w(0,\tau) = 0, & \tau \in \mathbb{R}_+ \\
w(\zeta,0) = -1, & \zeta \in \mathbb{R}_+ \\
w(s(\tau),\tau) = 0,\ w_x(s(\tau),\tau) + \dot{s}(\tau) = 0, & \tau \in \mathbb{R}_+ \\
s(\infty) = \sqrt{2}
\end{cases}
$$

the curve $\zeta = s(\tau)$ being the free boundary for the solution $w(\zeta,\tau)$ of (2.7.11).

One has also:

$$
(2.7.12) \quad \lim_{\tau\to\infty} v(\zeta,\tau) = (1-\zeta/\sqrt{2})_+^2.
$$

The proof of Theorem 2.7.1. and the claims stated in Remark 2.7.2, as well as the corresponding generalizations, will be presented elsewhere.

This research has been partially supported by the European Research Office of the US Army under the contract No. DAJA-37-82-C-0731.

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